IV–35. Stress Analysis of Brick Masonry as a "No Tension" Material

C. A. Anselmi — L. Fino
Istituto di Costruzioni — Facoltà di Architettura — Napoli (Italia)

ABSTRACT

A brick masonry cannot sustain tensions due to the material nature and the presence of joints. As a first step to problem we study the stress state for a "no tension" material, by a finite element formulation using an iterative "stress transfer" method. This approach, which has been studied for the rock mechanics using as unknowns the nodal displacements, here is used with a particular equilibrium model. The fundamental hypothesis is that no tensions can be alone on the interfaces. In this process, which bears some similarity with "relaxation" approaches, the fundamental steps are as follows:

a. the normal and the shear stresses will be found on the joints of the interfaces by the minimization of the complementary energy, assuming the continuum isotropic and linearly elastic;
b. the tensile stresses on the joints will be eliminated applying some "restraining" forces to the continuum;
c. the "restraining" forces will be removed by superposition of equal but opposite nodal forces: the continuum is then reanalyzed for the effect of such forces and the stresses computed will be added to those of the stage (b).

The steps (b) and (c) will be repeated until all tensile stresses will be reduced to a negligible quantity.

INTRODUCTION

An approach to develop the stress analysis of brick masonry considered as a "no tension material", discretized by finite elements and solved by stress method, is here retaken.

In the approximation, we limit ourselves to "linear theory" and to bi-dimensional elastic continuum.

The problem which we intend to consider is a "unilateral problem" whose formulation leads to inequalities: it is well known that the solution can be obtained studying a quadratic programming problem.

Here a nonstandard method, founded on the force method, is proposed: the basic hypothesis of stress linear variation has been adopted and, to avoid the consequent well known difficulties, a particular continuum discretization has been made.

Particularly, we have discretized continuum directly in triangular statically determinate "sub-regions", each one separated in three triangular elements, having a common vertex in the sub-region centroid (fig. 1).

In our method, which bears some similarity with "relaxation" approaches, the fundamental steps are as follows:

a. The finite element method is applied to continuum considered as having bilateral constraints. The element coupling procedure results in a 9×9 equation system which gives the element stresses as functions of the unknown stresses on the subregion boundary. The next subregion assembling procedure results in a statically indeterminate equation system; the stresses on the boundary are determinated associating the compatibility equation, obtained by the complementary energy stationarity conditions and the use of the Lagrangian multipliers, to the equilibrium equations.
b. The tensile stresses on the interfaces are eliminated applying some "restraining" forces on the same interfaces; the continuum is then reanalyzed for the effect of such forces and the stresses computed will be added to those of the stage (a).
c. The step (b) is repeated until all tensile stresses will be reduced to a negligible quantity.

This "stress transfer" method has been already proved to be always convergent.

FORMULATION OF THE PROBLEM

As we will see after, the stress state within any triangular element can be expressed by a vector of parametrical stresses:

$$\boldsymbol{\sigma}_e = \begin{bmatrix} \boldsymbol{\sigma}_e^1 \\ \boldsymbol{\sigma}_e^2 \end{bmatrix}$$

where $\boldsymbol{\sigma}_e^1$ will be the sub-vector of the stresses normal to the interfaces and $\boldsymbol{\sigma}_e^2$ the sub-vector of the shear stresses.

Because it is necessary to satisfy the continuity requirements on interfaces between the elements, one has to write two conditions for every interface e−h in every vertex i:

$$\sigma_{ei}^1 = \sigma_{hi}^1 \; ; \qquad \sigma_{ei}^2 = \sigma_{hi}^2 \tag{1}$$

with the condition:

$$\sigma_i^1 \geqslant 0 \tag{2}$$

as the material is assumed to be incapable of sustaining tensile stresses along the interfaces.

The compatibility conditions are assigned with this fundamental hypothesis: where a fracture appears in the material, it is possible to have some displacement between the interfaces alone along the normal direction to the interfaces, and these displacements have to be correspondent alone to a separation.

Thus, if E_0 is the functional obtained adding the equilibrium conditions to the complementary energy of the system, by the Lagrangian multipliers, we will write:

$$\frac{\partial E_0}{\partial \boldsymbol{\sigma}_i^2} = \mathbf{0} \tag{3}$$

for the shear stresses, while we will put:

$$\frac{\partial E_0}{\partial \boldsymbol{\sigma}_i^1} \geqslant \mathbf{0} \tag{4}$$

for the normal stresses.

Thus, we have to observe another fundamental condition for this problem:

$$\boldsymbol{\sigma}_i^{1*} \frac{\partial E_0}{\partial \boldsymbol{\sigma}_i^1} = 0 \tag{5}$$

Now, it is easy to note that our problem is a "linear complementarity problem": if **x** is defined in a n-dimensional Euclidean space and **A** is a given $n \times n$ matrix, **b** is a given n-vector, we have to find the vector which satisfies the conditions:

$$\begin{aligned} &\mathbf{x} \geqslant \mathbf{0}\,; \\ &\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b} \geqslant \mathbf{0}\,; \\ &\mathbf{x}^*\mathbf{y} = 0 \end{aligned} \tag{6}$$

Under the assumption that matrix **A** be symmetric, the problem admits two complementary variational formulations:

$$\begin{aligned} \delta\phi &= \delta\mathbf{x}^*(\mathbf{A}\mathbf{x}_0 + \mathbf{b}) \geqslant 0 \\ \delta\psi &= \delta\mathbf{x}^*\mathbf{A}\mathbf{x}_0 \geqslant 0 \end{aligned} \tag{7}$$

and, if we suppose that the matrix **A** be both symmetric and positive definite, the formulation (6) is equivalent to the following convex quadratic programming problems:

$$\begin{aligned} &\min \phi(\mathbf{x}) | \mathbf{x} \geqslant \mathbf{0} \\ &\min \psi(\mathbf{x}) | \mathbf{A}\mathbf{x} + \mathbf{b} \geqslant \mathbf{0} \end{aligned} \tag{8}$$

A unique solution will exist if and only if **A** is a positive definite matrix; the solution may be not unique if **A** is a positive semidefinite matrix.

THE STRESS ANALYSIS FOR BILATERAL CONSTRAINTS

The continuum is discretized in triangular "sub-regions", each one separated in three triangular elements, having a common vertex in the sub-region centroid (fig. 1).

Assuming linear variation of stresses, these in every elements are:

$$\begin{aligned} \sigma_{11} &= a_0 - b_2 x_1 + a_2 x_2 \\ \sigma_{12} &= b_0 + b_1 x_1 + b_2 x_2 \\ \sigma_{22} &= c_0 + c_1 x_1 - b_1 x_2 \end{aligned} \tag{9}$$

or better:

$$\boldsymbol{\sigma} = \mathbf{x}\mathbf{k}^c \tag{9'}$$

where $\mathbf{k}^c$ is a 7-component vector because the conditions of overall equilibrium have to be satisfied within any element.

If we write the (9') in the general vertex P of an element, we can obtain the normal stress σ^P_{pn} and the shear stress σ^P_{pt} for the side p and in P with:

$$\begin{aligned} \boldsymbol{\sigma}^P_p &= \begin{bmatrix} \sigma^P_{pn} \\ \sigma^P_{pt} \end{bmatrix} \\ &= \begin{bmatrix} \alpha_1^2 & 2\alpha_1\alpha_2 & \alpha_2^2 \\ \alpha_1\beta_1 & \alpha_1\beta_2 + \alpha_2\beta_1 & \alpha_2\beta_2 \end{bmatrix} \cdot \begin{bmatrix} \sigma^P_{11} \\ \sigma^P_{12} \\ \sigma^P_{22} \end{bmatrix} \\ &= \mathbf{S}_P \boldsymbol{\sigma}^P = \mathbf{S}_P \mathbf{x}^P \mathbf{k}^c \end{aligned} \tag{10}$$

where α_i, β_i $(i = 1,2)$ are the direction cosines of the normal and the tangent lines to p.

Finally, if we put:

$$\mathbf{t}^c = \begin{bmatrix} \boldsymbol{\sigma}^G \\ \boldsymbol{\sigma}^A_a \\ \boldsymbol{\sigma}^B_b \end{bmatrix} = \begin{bmatrix} \mathbf{I} & & \\ & \mathbf{S}_a & \\ & & \mathbf{S}_b \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x}^G \\ \mathbf{x}^A \\ \mathbf{x}^B \end{bmatrix} \cdot \mathbf{k}^c = \mathbf{S}^c \cdot \mathbf{k}^c \tag{11}$$

it follows:

$$\boldsymbol{\sigma} = \mathbf{x}(\mathbf{S}^c)^{-1}\mathbf{t}^c \tag{9''}$$

and the stress state in every point of the element can be expressed as a function of the seven stresses $\mathbf{t}^c$, taken as fundamental independent parameters.

The vector of the four normal and shear stresses in the vertex A and B of the side g external to a subregion (fig. 2) for (10) and (11) is:

$$\begin{aligned} \boldsymbol{\sigma}^c_g &= \begin{bmatrix} \boldsymbol{\sigma}^A_g \\ \boldsymbol{\sigma}^B_g \end{bmatrix} = \begin{bmatrix} \mathbf{S}_g & \\ & \mathbf{S}_g \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x}^A \\ \mathbf{x}^B \end{bmatrix} \cdot \mathbf{k}^c \\ &= \mathbf{S}^c_g(\mathbf{S}^c)^{-1}\mathbf{t}^c = \mathbf{U}^c\mathbf{t}^c \end{aligned} \tag{12}$$

For the continuity requirements on interfaces between the three elements, now we would have to write four conditions to establish the equality of the stresses on every side, and we would obtain a twelve equation system; but, if the correspondent parametric stresses of the interconnected elements have the same name, the twelve conditions will be identically satisfied.

Consequently the 21 $\mathbf{t}^b_0$ parameters for the generical stress state in the three elements of a subregion are reduced to nine independent parameters $\mathbf{t}^b$ only, by a Boolean matrix (21×9) E:

$$\mathbf{t}^b_0 = \mathbf{E}\mathbf{t}^b \tag{13}$$

where, (fig. 3),

$$(\mathbf{t}^b_0)^* = [(\mathbf{t}^{C1})^*(\mathbf{t}^{C2})^*(\mathbf{t}^{C3})^*], \quad (\mathbf{t}^b)^* = [(\boldsymbol{\sigma}^G)^*(\boldsymbol{\sigma}^A_a)^*(\boldsymbol{\sigma}^B_b)^*(\boldsymbol{\sigma}^C_c)]$$

The twelve stresses on the external sides of the generical subregion (fig. 4) for the (12) and (13) are:

$$\begin{aligned} \boldsymbol{\sigma}^b_e &= \begin{bmatrix} \boldsymbol{\sigma}^{C1}_g \\ \boldsymbol{\sigma}^{C2}_g \\ \boldsymbol{\sigma}^{C3}_g \end{bmatrix} = \begin{bmatrix} \mathbf{U}^{C1} & & \\ & \mathbf{U}^{C2} & \\ & & \mathbf{U}^{C3} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{t}^{C1} \\ \mathbf{t}^{C2} \\ \mathbf{t}^{C3} \end{bmatrix} \\ &= \mathbf{U}^b\mathbf{t}^b_0 = \mathbf{U}^b\mathbf{E}\mathbf{t}^b = \mathbf{W}^b_0\mathbf{t}^b \end{aligned} \tag{14}$$

The system (14) has 9 independent equations only, because we have to satisfy the equilibrium requirements for the stresses $\boldsymbol{\sigma}^b_e$; it can be write over again:

$$\boldsymbol{\sigma}^b = \mathbf{W}^b\mathbf{t}^b \tag{15}$$

Finally, it follows:

$$\mathbf{t}^{b} = (\mathbf{W}^{b})^{-1}\boldsymbol{\sigma}^{b} \qquad (16)$$

thus, the parametric stresses in a sub-region are functions of nine independent stresses applied on the boundary of the same sub-region.

When we proceed to assemble the sub-regions, to obtain equation systems very small, we have to take the nine independent parameters on every subregion opportunely: precisely the nine parameters have to be assigned in the same manner for every subregion—three on every side and every vertex—as showed in fig. 5, so one has to write two equations on every interface r and one equation on every external side s.

The equation system obtained is "iperstatic"; the stresses on the boundary of the subregions can be established associating the compatibility equations, obtained by the stationarity requirements for the complementary energy, by the Lagrangian multipliers.

The continuum energy E can be expressed by:

$$E = \sum_{b=1}^{N} \frac{1}{2}\boldsymbol{\sigma}^{b*}\mathbf{M}^{b}\boldsymbol{\sigma}^{b} = \frac{1}{2}\boldsymbol{\sigma}_{0}^{*}\mathbf{M}\boldsymbol{\sigma}_{0} \qquad (17)$$

The compatibility conditions are given by the functional:

$$E_0 = E + \Sigma_i\lambda_i f_i \qquad (18)$$

where $f^i = 0$ are the equilibrium conditions and the λ_i are the Lagrangian multipliers.

It is well known that the operator of the system formed by the stationarity conditions for E_0 and equilibrium conditions is symmetric.

THE STRESS ANALYSIS FOR THE UNILATERAL PROBLEM

It is supposed the existence of unilateral constraints alone between the interfaces of the triangular elements in a subregion and the subregion interfaces: precisely, it is assumed that alone shear stresses and compressive stresses can exist between those interfaces.

The stress state analysis for the continuum having bilateral constraints gives us the possibility to find where tensile stresses are developed. As the material is assumed incapable of sustaining tensions along interfaces, they are eliminated applying "restraining" forces.

Precisely, for the sub-region where the tensile stresses on the interfaces between the elementar triangles have the largest values, it is established the stress distribution on the subregion boundary—by the (15)—to whom a zero stress corresponds where tensile stresses were been found.

Thus the stress state connected to this first "restraining" force system is established for the same continuum, still considered elastic and with bilateral constraints, and it is added to the stress state for the external loads.

Then we will develop the same operation for another subregion where the tensile stresses have the largest values and then for all the others, until all tensile stresses are reduced to a negligible quantity.

Finally, we have to develop the same operations for the interfaces between the sub-regions, when tensile stresses are found on these.

If at the end of this third stage of our process, we will find tensile stresses still between the interfaces of the elementar triangles in some sub-region, we have to repeat all the operations before described.

It is known that this "stress transfer" method has been proved to be always convergent.

FINAL REMARKS

In this paper the demonstrations of existence, uniqueness and convergence of the solution are been omitted, because many others pages would have to be written and this was prohibited.

We preferred describing our process to furnishing those demonstrations, also because many others authors wrote about them.

According to us, the described method is not very difficult and is characterized by a good degree of accuracy, obviously with the limits imposed by the initial approximations.

Finally, it is very important that we have to study always the same model during the many steps of the process.

Now we are studying a particular bidimensional continuum by the exposed method and the results will be related to those of the others methods already known.

REFERENCES

1. Heyman J. (1966): "The stone skeleton". Int. J. Solids Struct. 2, 141–156
2. Zienkiewicz O. C. (1971): "The finite element method in engineering science". London, McGraw-Hill
3. Frémond M. (1973): in "The mathematics of finite elements and applications", 175–188. Witheman Ed. Academic Press.
4. C. A. Anselmi, E. De Rosa, L. Fino (1975): "Sul metodo delle forze con il procedimento degli elementi finiti" AIMETA, Cagliari
5. Duvaut G., Lions J. L. (1976): "Inequalities in mechanics and phisics". Berlin, Springer Verlag
6. Kunzi H. P., Krelle W. (1966): "Non linear programming" Blaisdell Publ. Co.

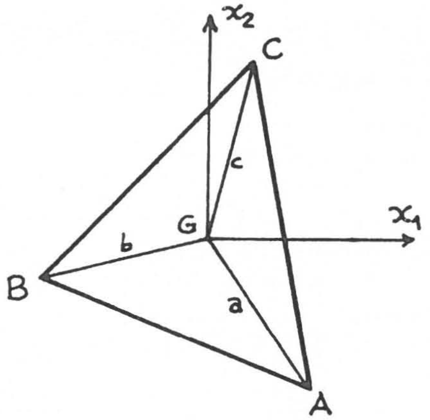

Figure 1.

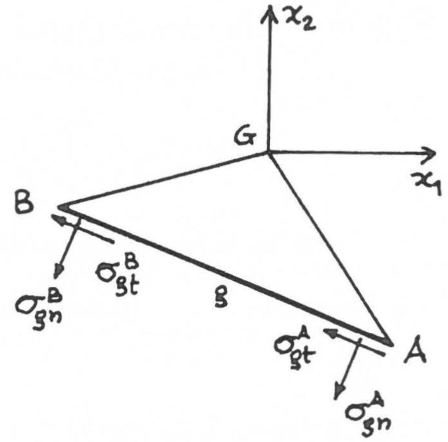

Figure 2.

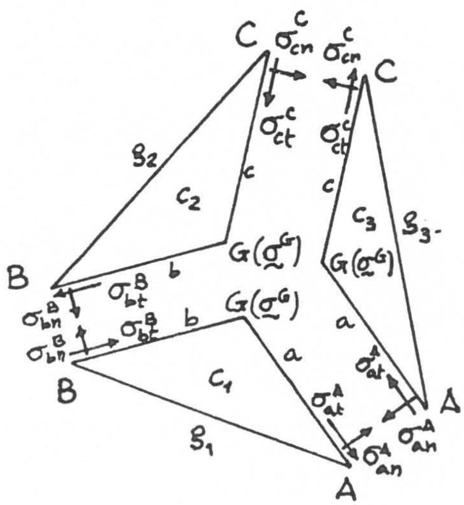

Figure 3.

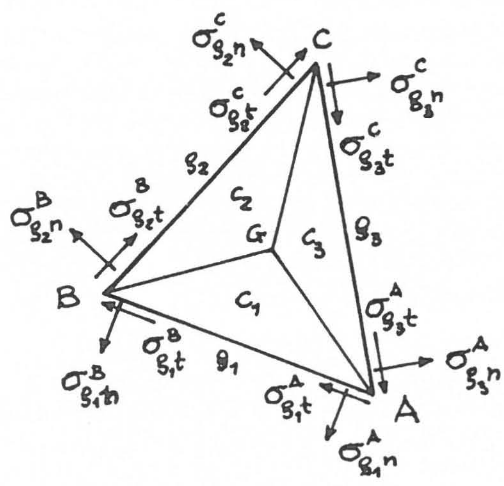

Figure 4.

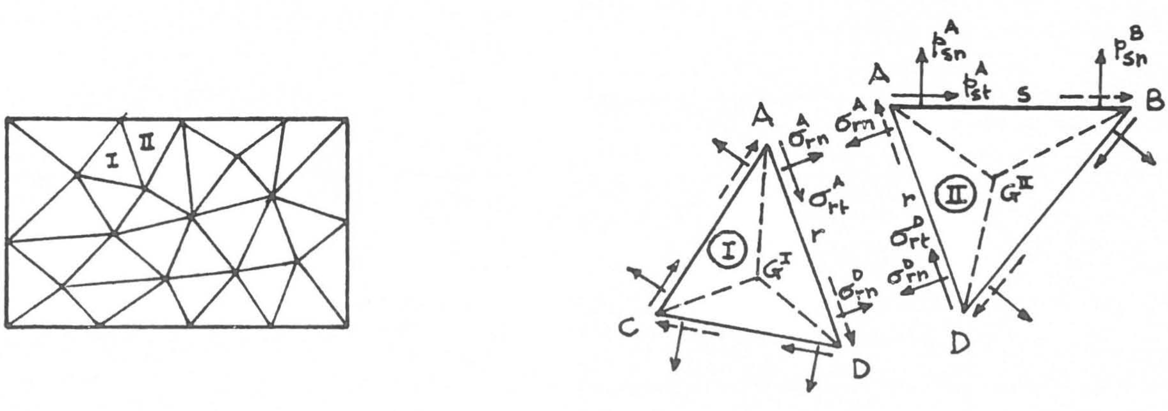

Figure 5.