On the Problem of Bending and Compression of Masonry Structures

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Abstract - The paper outlines a general theory of bending based on the concept of a limit state of unrestrained contact of a non monolithic structure. In this limit state where no tensile stresses occur at the joints the deformation parameters are subjected to certain extremum conditions which provide upper and lower bounds to the load carrying capacity of the structure. These parameters are determined by an analysis of the elastic contact problem at the cracked joint, paying attention to the tensile stresses in the block. Calculations carried out display a considerable effect of the relative block height on the overall stiffness and loadcarrying capacity of the structure.

1. The deformation parameters of the block and the joints

The deformations of the blocks may be determined by considering an infinite beam with constant crack spacing l, subjected to a normal force $N = -P$ and a constant bending moment $M = Pe$. This load will induce in the beam a periodic state of stress and strain $\{\sigma, \varepsilon\}$ where the contact sections a_i and the middle sections b_i due to the symmetry of loading will remain plane (Fig. 1). Hence

$$(1.1) \qquad u_x(\pm \tfrac{1}{2}, y, z) = \pm \tfrac{1}{2}(v + \omega y); \quad u_x(0, y, z) = \pm \tfrac{1}{2}\, \gamma(y, z)$$

The state of stress, strain and displacement may be decomposed

$$(1.2) \qquad \{\sigma, \varepsilon\} = \{\sigma_\infty, \varepsilon_\infty\} + \{\sigma_h, \varepsilon_h\}$$

$$(1.2') \qquad \{u\} = \{u_\infty\} + \{u_h\}$$

into the states of stress, strain and displacement of the uncracked beam $(l = \infty)$ and the states $\{\sigma_h, \varepsilon_h\}$ and $\{u_h\}$ caused by the edge effect of the crack where $\sigma_x = 0$ and the crack width $\gamma > 0$.

The extension v of the period $b_{\nu-1} - b_{\nu}$ (Fig. 1b) at the level of the elastic centroid 0, the shortening u_p at the level of the normal force $N = -P$ and the rotation angle ω of the end faces b are determined with the aid of Betti's rule and work equations

$$v = v_{\infty} + v_h = -\frac{Pl}{EA} + \frac{1}{A}\int \gamma dA$$

$$\omega = \omega_{\infty} + \omega_h = \frac{Pel}{EI} + \frac{1}{I}\int \gamma y dA \qquad (1.3)$$

$$u_p = u_{p\infty} + u_{ph} = \frac{Pl}{EA}\left(1+\left(\frac{e}{i}\right)^2\right) + \frac{e}{I}\int \gamma y dA$$

As the first terms on the right give the generalized deformations in the uncracked state the second terms represent the generalized deformation of the edge effect. The crack width $\gamma(y,z)$ may be replaced by a geometrically equivalent linearized discontinuity distribution (Fig. 1c)

$$(1.4) \qquad \bar{\gamma} = [u_x]_0 = v_h + \omega_h y \ ; \quad \left(\int \gamma dA = \int \bar{\gamma} dA \ ; \ \int \gamma y dA = \int \bar{\gamma} y dA\right)$$

the portion of the beam between the cracks being deformed according to the conventional theory of bending (u_{∞}, ε_{∞}). Writing

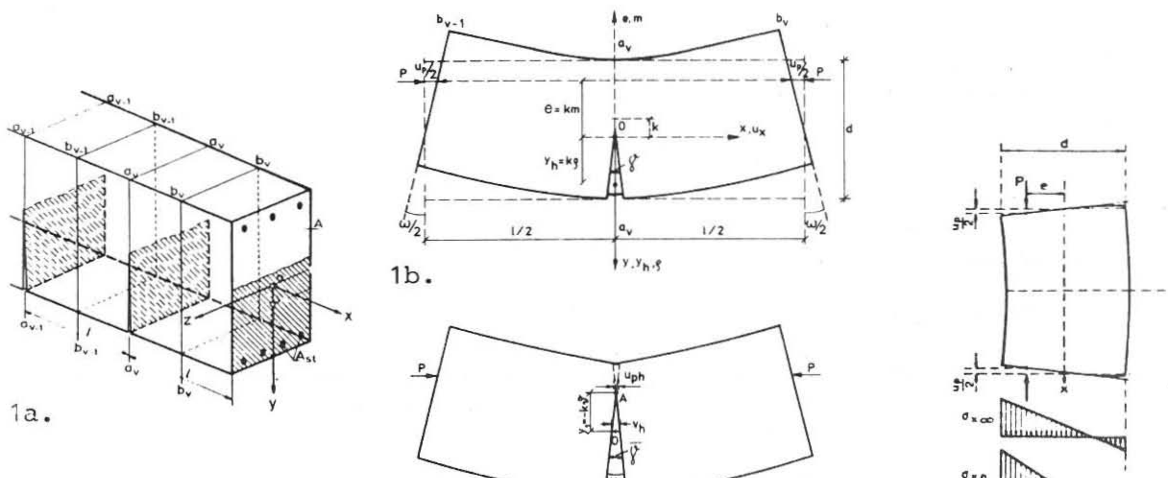


Fig. 1 - a) Cracked beam b) Real deformation c) Linearized deformation d) $\sigma_{x\infty}$, σ_{xo}.

$$u_p = \frac{Pd\delta}{EA}\ ;\quad u_{p\infty} = \frac{Pd\delta_\infty}{EA} = \frac{Pd\lambda}{EA}\left(1+\left(\frac{km}{i}\right)^2\right)\ ;\quad u_{ph} = \frac{Pd\delta_h}{EA}$$

$$(1.5)\qquad v = \frac{Pd\beta}{EA}\ ;\quad v_\infty = \frac{Pd\beta_\infty}{EA} = -\frac{Pd\lambda}{EA}\ ;\quad v_h = \frac{Pd\beta_h}{EA}$$

$$\omega = \frac{Pd\alpha}{EAk}\ ;\quad \omega_\infty = \frac{Pd\alpha_\infty}{EAk} = \frac{Pd\lambda m}{3EAk}\ ;\quad \omega_h = \frac{Pd}{EAk}\alpha_h$$

The parameters α, β, λ depend on the relative eccentricity $m = e/k$ of the force P and the ratio $\lambda = 1/d$.

Because $\delta(\lambda, m) = u_p(\lambda, m)/u_p(\lambda, 0)$ this parameter is directly related to the compressive stiffness D and the strain energy W of the block

$$(1.6)\qquad D(\lambda m) = P/u_p = \frac{EA}{d\delta(\lambda m)}\ ;\quad W = \frac{Pu_p}{2} = \frac{P^2 d\delta(\lambda m)}{2EA}$$

In working conditions a cracked beam finally attains a limit state of unrestrained contact where by reason of repeated loading cycles at the cracked joints no tensile stresses occur.

In this special case for δ are obtained lower (δ_ε) and upper (δ_σ) bounds by the extremum principles of stiffness [1] [2]. According to them the block stiffness D:

a) if defined by a permissible equilibrium state of stress $\{\sigma'\}$ where at the joint $\sigma_x \leqq 0$

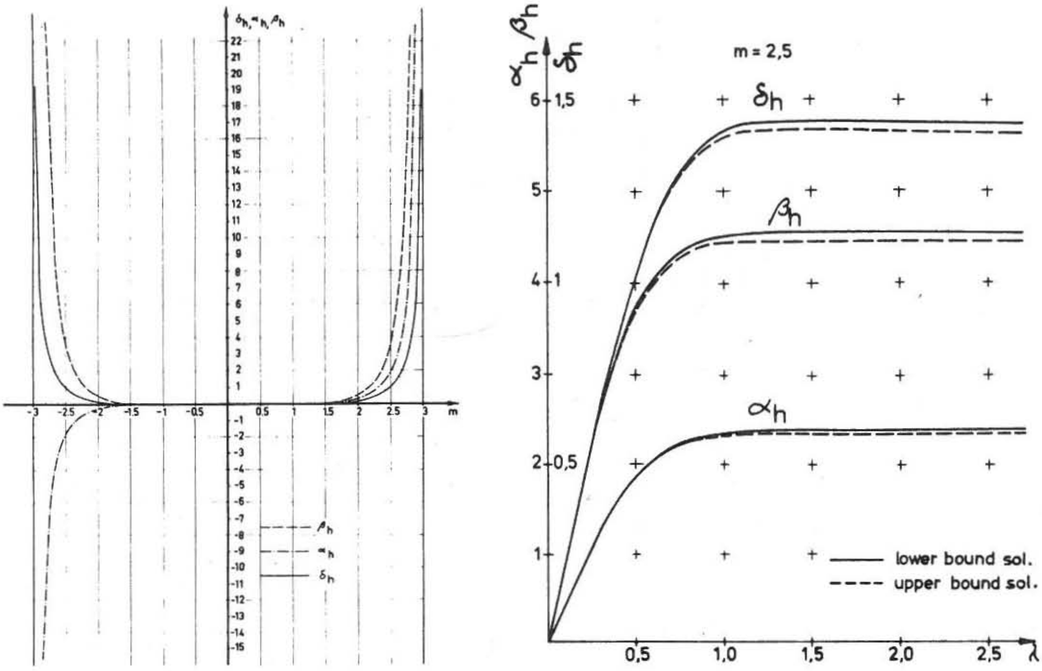
$$(1.7)\qquad D'_\sigma = \frac{P^2}{2\ W(\sigma')}$$

attains a maximum in the actual state $\{\sigma\}$, which corresponds to a permissible state of deformation.

b) if defined by a permissible state of deformation $\{u'', \varepsilon''\}$ where $\gamma'' \leq 0$

$$(1.7')\qquad D''_\varepsilon = \frac{2\ W(\varepsilon'')}{(u''_p)^2}$$

attains a minimum in the actual state, which corresponds to a permissible state of equilibrium.



2a. 2b.

Fig. 2 - The functions $\delta_h(\lambda,m)$, $\alpha_h(\lambda,m)$, $\beta_h(\lambda,m)$. a) $\lambda=4$ b) $m=2.5$

The strain energy of the period can be decomposed

$$(1.8) \qquad W(\sigma,\varepsilon) = W(\sigma_\infty,\ \varepsilon_\infty) + W(\sigma_h, \varepsilon_h)$$

Because of the plane contact faces

$$(1.9) \qquad \frac{\partial W(\sigma)}{\partial M} = \omega \ ; \quad \frac{\partial W(\sigma_h)}{\partial M} = \omega_h \ ; \quad \frac{\partial W(\sigma)}{\partial N} = v \ ; \quad \frac{\partial W(\sigma_h)}{\partial N} = v_h$$

Hence the following relations hold (Fig. 1c)

$$(1.9') \qquad \alpha = \frac{1}{2}\frac{\partial\delta}{\partial m} \ ; \quad \beta = \frac{m}{2}\frac{\partial\delta}{\partial m} - \delta \ ; \quad \alpha_h = \frac{1}{2}\frac{\partial\delta_h}{\partial m} \ ; \quad \beta_h = m\alpha_h - \delta_h \ ;$$

$$(1.10) \qquad \frac{|\alpha_h|}{\beta_h} = \frac{k^2}{i^2}|\rho| \ ; \quad |\rho\upsilon| = (i/k)^2 \ ; \ (\text{if no reinforcement } \frac{\alpha_h}{\beta_h} = \rho/3)$$

$$\beta_h = m\alpha_h - \delta_h \geqq 0 \ ; \quad \delta_h = (m-\upsilon)\alpha_h \geqq 0 \ ; \quad |m| > |\upsilon| \ ;$$

$\mathrm{sign}\,\alpha_h = \mathrm{sign}\,m = \mathrm{sign}\,\upsilon$ where $\rho = y_h/k$; $\upsilon = -y_1/k$; $y_1 = -y_h/\omega_h$.

The expressions

$$(1.11) \qquad \dot{u}_p = \frac{u_p}{l} = \frac{P}{EA}\dot{\delta} \ ; \quad \kappa = \frac{\omega}{l} = \frac{P}{EAk}\dot{\alpha} \ ; \quad \dot{v} = \frac{v}{l} = \frac{P\dot{\beta}}{EA}$$

define the average strain $\dot{u}_p$ at the level y = -e, the average curvature κ and the strain $\dot{v}$ of the centroid axis, with $\dot{\delta} = \delta/\lambda$; $\dot{\alpha} = \delta/\lambda$; $\dot{\beta} = \beta/\lambda$. In the cases of plane strain and plane stress the parameters α, β, δ are identical. The solution is known in two limit cases only (Fig. 1d)

a) (1.12) $\lambda = \infty$: Stresses : $\sigma_x = \frac{P}{A}(-1+m\eta)$; $\sigma_{y\infty} = \tau_{xy\infty} = 0$
$(\eta = 2y/d)$

Deformation parameters

(1.12') $$\dot{\delta}_\infty = \dot{\delta}(\infty, m) = 1+\frac{m^2}{3} ; \quad \dot{\alpha}_\infty = \frac{m}{3} ; \quad \dot{\beta}_\infty = -1$$

b) $\lambda = 0$: No tensile stresses

(1.13) $$\sigma_{xo} = \frac{4P}{A} \frac{(\eta-2+m)}{(3-|m|)^2} ; \quad \sigma_{yo} = \tau_{yxo} = 0 ;$$

Deformation parameters

(1.13') $$\dot{\delta}_o(m) = \dot{\delta}(o,m) = \frac{8}{3(3-|m|)} ; \quad \dot{\alpha}_o = \frac{4 \operatorname{sign} m}{3(3-|m|)^2} ;$$
$$\dot{\beta}_o = \frac{4(|m|-2)}{(3-|m|)^2}$$

(1.13") $$\dot{\delta}_{ho} = \dot{\delta}_o - \dot{\delta}_\infty = \frac{(|m|-1)^3}{3(3-|m|)^2} ; \quad \dot{\alpha}_{ho} = \frac{((|m|-1)^2(4-|m|)}{3(3-|m|)^2} \operatorname{sign} m;$$
$$\dot{\beta}_{ho} = \frac{(|m|-1)^2}{(3-|m|)^2}$$

For the intermediate cases $0 < \lambda < \infty$ a lower bound solution is sought by a stress function

(1.14) $$\Phi = f(y) + \sum_{i=1}^{n} g_i(y)\Psi_i(x)$$

where f(y) satisfies the stress conditions at the joints (Fig. 3a)

(1.15) $$\frac{\partial^2 f}{\partial y^2} = \sigma_x(\pm \tfrac{1}{2}) < 0, \text{ if } y < y_o ; \quad \frac{\partial^2 f}{\partial y^2} = 0, \text{ if } y > y_o$$

y_o coordinate of cracktip

(1.15') $$b \int_{-d/2}^{+d/2} \frac{\partial^2 f}{\partial y^2} dy = P; \quad b \int_{-d/2}^{+d/2} \frac{\partial^2 f}{\partial y^2} y dy = Pe$$

The functions g_i and ψ_i satisfy the conditions

(1.15") $(\frac{\partial g}{\partial y})_{\pm\frac{d}{2}} = 0$; $g(\pm \frac{d}{2}) = 0$; $(\frac{\partial \psi}{\partial x})_{\pm\frac{1}{2}} = 0$; $\psi(\pm \frac{1}{2}) = 0$;

$$\psi(-x) = \psi(+x)$$

corresponding to the boundary condition $\tau_{xy} = 0$ on the conture and $\sigma_y(x, \pm \frac{d}{2}) = 0$. In this case the solution is obtained by minimizing the functional

$$U = \frac{1}{2E} \int_{-1/2}^{+1/2} \int_{-d/2}^{+d/2} \Delta\phi^2 dxdy + \mu_1 (b\int \frac{\partial^2 f}{\partial y^2} dy - P) + \tag{1.16}$$

$$+ \mu_2 (b \int \frac{\partial^2 f}{\partial y^2} ydy - Pe)$$

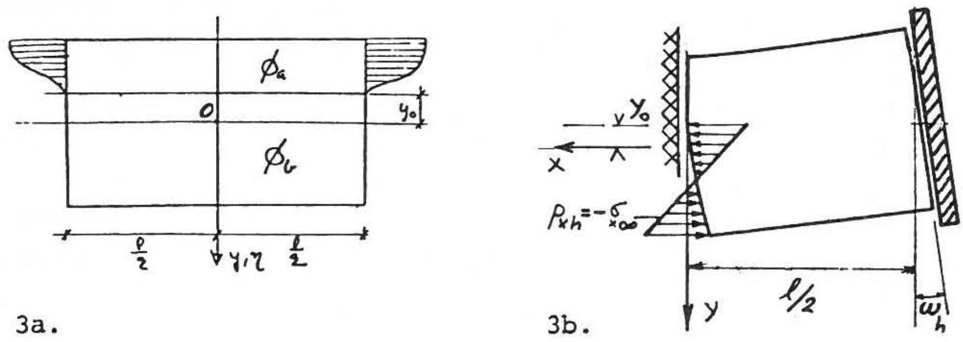
A suitable solution is obtained by choosing $\psi_i = (1-(-1)^i \cos 2i\pi x/l)$. f is an unknown function satisfying (1.15). Each g_i comprises two unknown functions $g_{ia}(y)$ for $y \leq y_o$ and $g_{ib}(y)$ for $y \geq y_o$ which satisfy continuity conditions $[g]_{yo} = g_b(y_o) - g_a(y_o) = 0$; $[\partial g/\partial y]_{yo} = 0$ (Fig. 3a).

The unknown functions f, g_{ia}, g_{ib} and the value y_o are determined by the calculus of variation applied to U. No closed expressions are obtained for the deformation parameters which are calculated numerically.

An upper bound solution based on a kinematically permissible state is obtained by considering a clamped cantilever (Fig. 3b) loaded at the cracked base by the known load in the domain $\frac{d}{2} \leq y \leq y_o$ $p_{xh} = -\sigma_{xh} = -(\sigma_x - \sigma_{x\infty}) = -\sigma_{x\infty}$. Because the deformation parameters $\delta, \alpha, \beta, \delta_h, \alpha_h, \beta_h$ do not depend on Poisson's ratio the parameters are obtained from the principle of minimum potential energy π_h with $\nu = 0$.

$$\pi_h = \frac{bE}{2} \int_0^{1/2} \int_{-d/2}^{+d/2} (\varepsilon_{xh} + \varepsilon^2_{yh} + \frac{1}{2}\gamma^2_{xyh}) dxdy - b\int_{y_o}^{d/2} p_h u_{xh} dy \tag{1.17}$$

A solution is sought by Ritz method using a state of displacements $\{u_x u_y\}$ expressed by a sum of double Fourier expansions in directions x and y and simple Fourier expansion in direction x with co-efficients comprising known exponential crack width functions $\gamma_i(y) > 0$.



3a. 3b.

Fig. 3 - Boundary conditions used a) force method b) displacement method.

The displacement method gives lower bounds to the deformation parameters whereas the force method provides upper bounds (Fig. 2b). A comparison between the force method and the displacement method indicates that this difference is kept within 2.5 % if $m \leqq 2.5$.

The dependence on λ is governed by two asymptotes: If $\lambda \to 0$

$$\delta_{ho}(m) = \lambda\dot{\delta}_{ho}(m) \ ; \quad \alpha_{ho} = \lambda\dot{\alpha}_{ho}(m); \quad \beta_{ho} = \lambda\dot{\beta}_{ho}(m) \text{ (formulae 1.13")}$$

If $\lambda \to \infty$ the parameters approach values independent of λ

$$\delta_h(\infty,m) \ ; \quad \alpha(\infty,m) \ ; \quad \beta_h(\infty,m)$$

These asymptotes provide good approximations of the parameters in the ranges $0 \leq \lambda \leq 0{,}3$ and $\lambda > 1$ respectively. The force method provides numerically more stable values. These may be approximated by the function

$$(1.18) \qquad \delta_h(\lambda,m) \cong \lambda\dot{\delta}_{ho}(m)\left(1+\frac{(1-(|m|-2)^2)\lambda^4}{6.15+\lambda^9}\right)\left[1-e^{-\left(\frac{\sqrt{r}}{\lambda}\right)^{\frac{1}{r}}(1-(m-2)^2)^r}\right]$$

where $r = 0.537$. If $\lambda \to \infty$

$$(1.18') \qquad \delta_h(\infty,m) \cong \sqrt{r}\,(1-(|m|-2)^2)^r\dot{\delta}_{ho}$$

The calculations show that the distance between the centroid of the stressblock and the centroid of the crackvolume is almost constant $\frac{2d}{3}$ or

$$(1.19) \qquad m + \rho \cong 4$$

2. Theory of Bending and Compression

The results of the previous section corresponding to constant eccentricity may be extended to the more general case with variable eccentricity

$$(2.1)\qquad m(x) = \frac{M(x)}{P} = m(o) + \frac{(m(L)-m(o))x}{L} + \frac{M_p(x)}{Pk} + \frac{w(x)}{k}$$

where M_p is the bending moment of the transverse load p of the simply supported beam with span L and w is the deflection of the beam.

A) If the number of the blocks is small it is assumed that deformation parameters of the joint δ_h, β_h, α_h only depend on the local eccentricity $m_i = m(x_i)$ at the joint i

$$\delta_{hi} = \delta(\lambda, m_i)\ ;\quad \beta_{hi} = \beta_h(\lambda m_i)\ ;\quad \alpha_{hi} = \alpha(\lambda m_i)$$

The axial strain v' and the curvature $\kappa = -w''$ of the block are (Fig 1c):

$$(2.2a)\qquad v' = v'_\infty = \frac{\partial \dot{W}_\infty}{\partial N} = \frac{N}{EA}\ ;\quad w'' = w''_\infty = -\frac{\partial \dot{W}_\infty}{\partial M} = -\frac{M}{EI}$$

Additionally every joint i induces (Fig. 1c):

a) a discontinuity of the slope of the deflection w

$$(2.2b)\qquad [\frac{\partial w}{\partial x}]_i = (\frac{\partial w}{\partial x})_+ - (\frac{\partial w}{\partial x})_- = -\omega_{hi} = -\frac{\partial W_{hi}}{\partial M} = -\frac{P_i d\alpha_{hi}}{EAk}$$

b) a discontinuity of the axial displacements u_x^o at the level of the elastic centroid

$$(2.2c)\qquad [u_x^o]_i = u_{x+}^o - u_{x-}^o = v_{hi} = \frac{\partial W_{hi}}{\partial N_i} = \frac{P d\beta_{hi}}{EA}$$

The strain energy of the structure is

$$(2.2c)\qquad W = W_\infty + W_h = \frac{1}{2}\int_o^L (\frac{N^2}{EA} + \frac{M^2}{EI})dx + \sum_i^n \frac{P_i^2 d\delta_{hi}}{2EA}$$

B) The number of blocks is great (n > 4). In this case the

discontinuities are smoothed out by using medium parameter values $\dot{\delta}$, $\dot{\alpha}$, $\dot{\beta}$. The strain energy per unit length

$$(2.3) \qquad \dot{W}(x) = \dot{W}_\infty + \dot{W}_h = \frac{P^2}{2EA}\,\dot{\delta}(m) \; ; \quad \dot{\delta} = 1 + \frac{m^2}{3} + \dot{\delta}_h$$

The total strain energy

$$(2.3') \qquad W = \int_0^L \dot{W}dx = \frac{1}{2}\int_0^L \left(\frac{N^2}{EA} + \frac{M^2}{EI}\right)dx + \int_0^L \frac{P^2\dot{\delta}_h}{2EA}dx$$

The deformations are denoting $\frac{\partial f}{\partial x} = f'$

$$(2.4a) \qquad v' = \frac{\partial \dot{W}}{\partial N} = \frac{P\dot{\beta}}{EA} = \frac{N}{EA} + \frac{P\dot{\beta}_h}{EA}$$

$$(2.4b) \qquad w'' = -\frac{\partial \dot{W}}{\partial M} = -\frac{P\dot{\alpha}}{EAk} = -\frac{M}{EI} - \frac{P\dot{\alpha}_h}{EAk}$$

Here N, P, M are functions of the axial coordinate x and $\dot{\delta}$, $\dot{\alpha}$, $\dot{\beta}$ are functions of m(x) only.

The general solution is highly dependent on the boundary conditions. If the expansion is restrained the dilatations of the cracks give rise to a considerable arching effect and the solution depends on $\dot{\alpha}$ and heavily on $\dot{\beta}$.

Thus if the supports are unmovable the thrust developed at a uniform transverse load p exceeds the minimum value

$$(2.5) \qquad P_{min} = \frac{pL^2}{8d}$$

This minimum is acc. to formula (1.19) attained when the deflections are small and l = L, i.e. cracks occur at the clamped supports only [2].

If the expansion of the axis is not restrained the solution depends only on $\dot{\alpha}$.

If the lateral dimensions of the block, the compressive force P and the transverse load p remain constant P', p' = 0 then acc. to (2.1) and (2.4b).

$$(2.5) \qquad w'' = km'' + p/P \; ; \quad \frac{\partial \dot{W}}{\partial M} = \frac{1}{M'}\frac{d\dot{W}}{dx}$$

Hence

$$(2.5') \qquad m'' + p/Pk = -\frac{1}{Pk^2m'}\frac{\partial W}{\partial x} \Rightarrow \frac{d}{\partial x}\left(\frac{k^2m'^2}{2} + \frac{\dot{W}}{P} + \frac{pkm}{P}\right) = 0$$

and

$$(2.6) \qquad \frac{k|dm|}{\sqrt{C-2(\dot{W}+pkm)/P}} = |dx|$$

from which by integrating and inserting the values of $\dot{W}$ and end conditions the maximum values of P can be determined by methods analogous to those developed in [3]. In the case of eccentric compressive loading only ($p = 0$) when the eccentricity attains a maximum m_n in the span then $C = \frac{P\dot{\delta}(m_n)}{EA}$ and the buckling load is obtained from the formula

$$(2.7) \qquad P_{cr}(\lambda, m_o) = \frac{EI}{L^2}\left[\frac{\mu(m_n, m_o) + \mu(m_n, m_L)}{2}\right]^2_{max}$$

where $\mu(m_n, m_i) = \frac{2}{\sqrt{3}}\int_{m_i}^{m_n}\frac{dm}{\sqrt{\dot{\delta}(\lambda m_n) - \dot{\delta}(\lambda, m)}}$ depends only on the medium stiffness parameter $\dot{\delta}(\lambda, m)$ of the block. In the case of equal end eccentricities $m_L = m_o$

$$(2.7') \qquad P_{cr}(\lambda, m_o) = \frac{EI}{L^2}\mu^2(m_n m_o)_{max}$$

Calculations carried out by using stiffness parameters $\delta(\lambda m)$ of the force method which provide true lower bounds to the critical load reveal that the block height ratio λ decisively increases the buckling load if $\lambda > 1$ and $m_o \geq 1.2$ (Fig. 4). The block height also increases the deflection at the critical load. On Fig. 5 load-deflection curves for various λ-ratios calculated according to the continuous scheme B) are compared with the deflection curve of a column consisting of two slender blocks calculated according to the discontinuous scheme A). Tests with steel block columns seem to confirm these theoretical findings [4].

The use of overhigh blocks is limited by the comparatively low modulus of rupture f_r of the block. It depends on the compressive strength f_m of the masonry, the maximum compressive stress σ_c at the joint and the maximum bending tensile stress σ_t of the block.

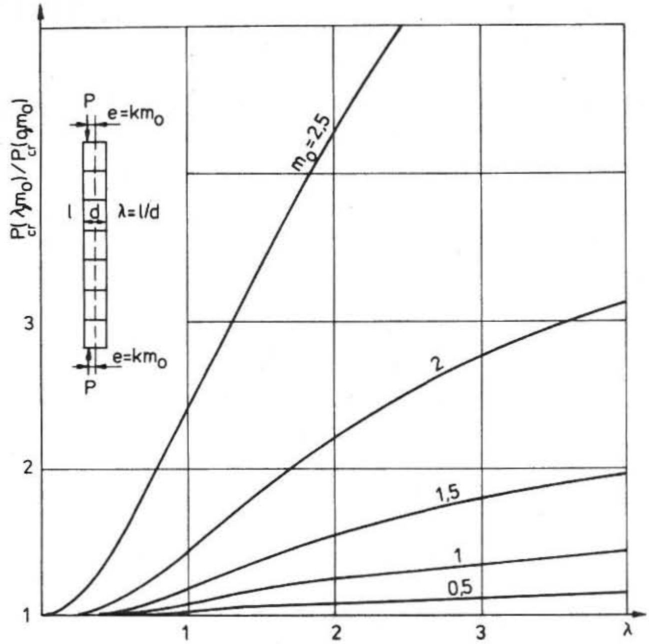


Fig. 4 - Effect of the block height ratio λ on the buckling strength ratio $P_{cr}(\lambda,m_o)/P_{cr}(0,m_o)$.

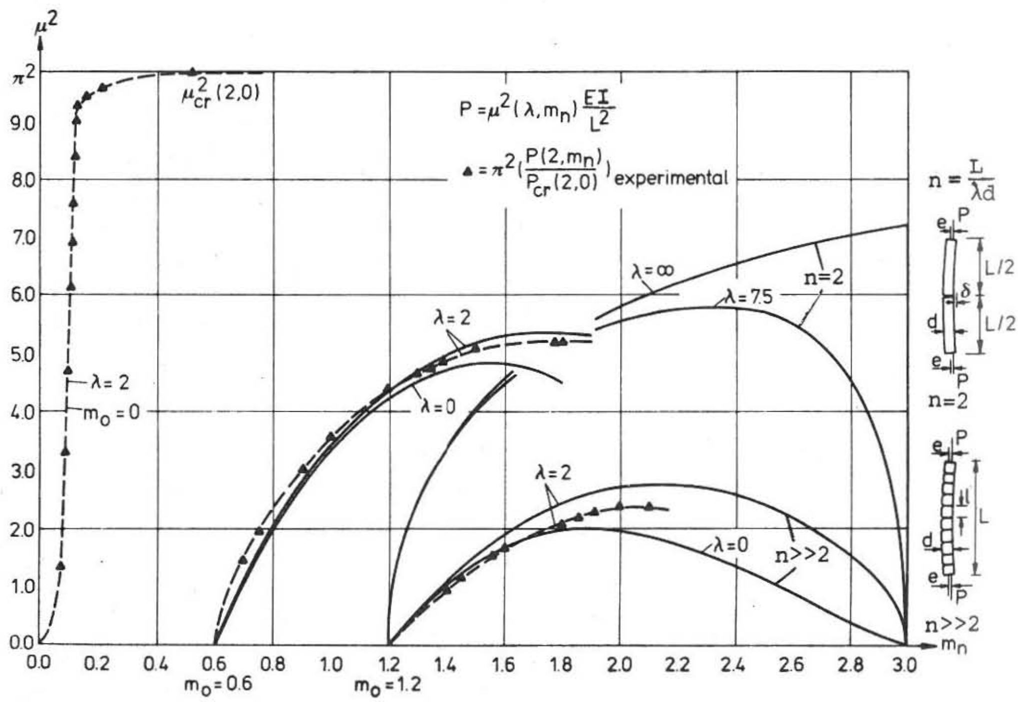


Fig. 5 - Theoretical — and experimental-▾-load deflection curves of columns.

According to the elastic analysis of the dry joint σ_c is almost independent of the block height ratio λ. The deviation from the maximum stress σ_{co} of the triangular distribution ($\lambda = 0$ formula 1.13)

$$(2.8) \qquad \sigma_{co} = -\frac{4}{3-m_n}\frac{P}{A}$$

does not exceed 3 %. The maximum tensile stress σ_t does not exceed the value 0.4P/A if $\lambda < 0.9$. With increasing λ it approaches the max. value of formula (1.12)

$$(2.8') \qquad \sigma_t = \sigma_{t\infty} = (m_n-1)P/A$$

Therefore the ratio

$$(2.8'') \qquad \sigma_t/|\sigma_{t\infty}| \cong \frac{1-(m_n-2)^2}{4} \leqq 0.25$$

attains it maximum at $m_n = 2$. Hence if $f_r/f_m \geqq 0.25$ crushing will be decisive and if the critical strength $\sigma_{cr} < f_r/2$ elastic buckling will be decisive. In these both cases no cracking will occur in the block and the strength increasing effect of the block height ratio λ can be utilized to the full.

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Notations

d,k	depth, kern distance of section	A	area of cross-section
l	height of block, crack spacing	$I = i^2A$	second moment of area
m = e/k	relative eccentricity	L	span, height of column
u,w	displacement, deflection	$\lambda = l/d$	height-depth ratio of block