Some Statistical Aspects of Masonry data

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Abstract - Most masonry properties and characteristics are variables. Statistical analysis of the variability of these characteristics should improve estimations of the values of properties, and the associated confidence in those values. For example the probability of failure of a masonry structural element will depend on the interplay and variability of several factors. Experimental data for strength, elastic modulus and creep are examined, and it is shown that these properties can be well represented by normal, lognormal and gamma distributions. Application of linear regression to the relationship between strength and elastic properties using classical and Bayesian approaches is shown. Finally, a numerical example dealing with the control of deflections is given.

1. INTRODUCTION

Engineering design almost inevitably involves relying on parameters whose actual values and characteristics are only known to a limited extent. In the design of masonry structures, strength, elastic, shrinkage and creep characteristics of the material have to be incorporated. Pre-design and site tests can only yield limited quantities of data. The designer has to make a decision based on this limited amount of information: such decisions should be arrived at through interpreting the data in a manner consistent with sound statistical theory.

Most engineering characteristics of materials may be considered to be random quantities. Many design problems involve manipulating functions of a number of these random quantities. For example, deflections depend on the elastic, shrinkage and creep characteristics, all of which are random quantities. It is important, therefore, to study the variability embodied in experimental records so as to be able to estimate probability distributions for the random quantities of interest. There is also the intriguing problem of how to use test data to modify one's prior estimate or "knowledge" of a particular characteristic.

We adopt the term "random quantity" (as suggested by de Finetti, [1]) rather than "random variable". Consider the following problem. We are designing a masonry element and wish to estimate its deflection. At the time of design we are uncertain about its elastic modulus: we express our uncertainty by means of a probability distribution. Once built, we

could measure accurately the elastic modulus: it is not a variable at all. In this problem we would therefore term the elastic modulus (at the time of design) as a "random quantity". To call it a "random variable" would suggest that it varies in a random way, which is not true. In the same vein, we do not believe that probability is a frequency: one should naturally use frequencies to evaluate a probability, provided that this is appropriate. Frequencies are our records of the past; probabilities generally in engineering deal with our uncertainty regarding future events. The connection between probability and frequency is given in a theorem by de Finetti [1]; if we consider a future event and a set of past events to be equi-probable, then the probability is the expected value of the frequency. The latter would be close to an observed frequency if we have a large number of observations.

In general the following rules apply: (a) A probability distribution is not a property of, or "attached to" the object of interest. Rather it describes the state of knowledge regarding the quantity of interest. (b) Before using frequencies to estimate probabilities, it is necessary that the events of a similar kind be judged to be equi-probable. This judgement might require further information, for example, the source and composition of masonry units: this point requires introspection, common sense, and the willingness to change as new information becomes available. (c) It follows that once a probability is evaluated by whatever method the probability should not be "cast in stone". Probability distributions should be revised as the state of information changes and should reflect factors such as new manufacturing techniques, materials, etc.

2. VARIABILITY IN RECORDED MASONRY PROPERTIES

A survey of published masonry test results shows that there is wide variability in masonry properties and that in general the variability is higher than for corresponding concrete properties. The following are a sample of the values of the coefficients of variation which have been reported:

Compressive strength:

Brick:	8.9 - 24%	(Maurenbecher [2])
Brickwork pier:	13.4 - 36%	(Beach and West [3], Lawrence and Morgan [4])

Elastic modulus:

Brickwork prism:	20 - 33%	(Lawrence and Morgan [4])

These figures show that the properties are random quantities which can take on values over a large range, and the large coefficients of variation indicate that the probability distributions should be rather flat.

In structural design the aim is to produce a member or section which has sufficient strength and rigidity to withstand safely the anticipated loads. The decision on the size of a section which can support a given magnitude of load will depend on two inter-related items, namely, the amount of risk the designer (or code-writing committee) is willing to take, and the level of uncertainty which is associated with the values of material and section strengths. Let us assume the following notation:

Q = load on a structure or structural element, or effects of a load (eg. stress, deflection).

R = resistance capacity of a structure (eg. strength) or the maximum "safe or acceptable" value of the effect of a load (eg. crack width).

Deterministic design philosophies have embodied the assumption of spiked distributions for Q and R (Fig. 1a) and have insisted that

$$R - Q > S$$

whereas S = the safety margin.

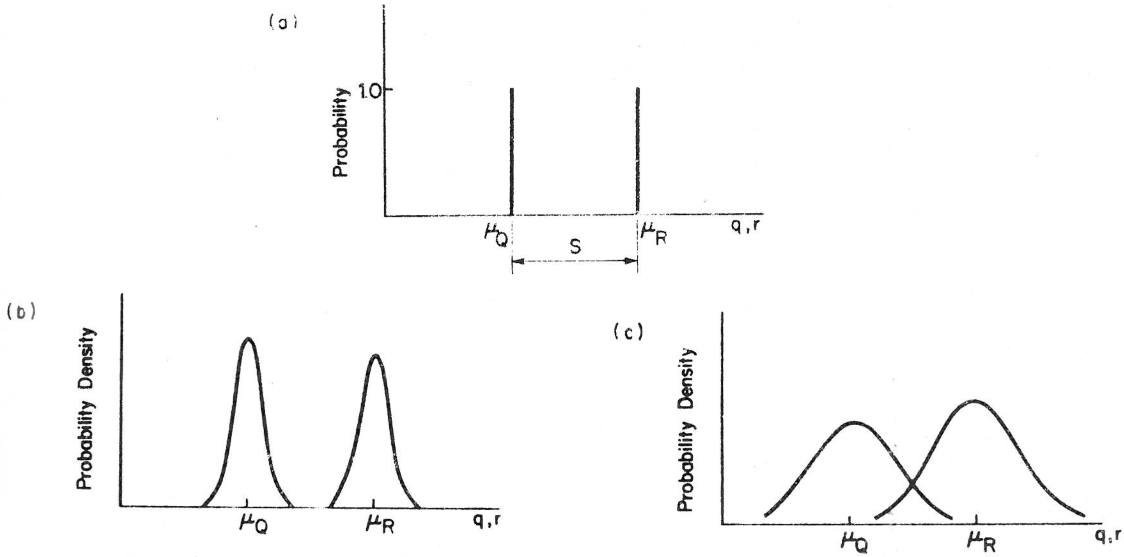

Fig. 1 Probability distributions of load Q and resistance R

Q and R are simply estimates based on experimental and theoretical data and experience. This kind of approach can be seen as a design strategy, or as an approximation if the variances of Q and R are small (Fig. 1b). From the test reports above it is clear that in masonry the situation is as repsresented by Fig. 1c.

In the cases shown in Fig. 1a, b it is assumed that provided the means μ_Q and μ_R are known and $S = \mu_R - \mu_Q$ is sufficiently large, that the probability of "failure"* is zero. In Fig. 1c the overlap of the distributions indicates that it is possible for the strength to be less than the load, and therefore there is a non-zero probability of failure. The overlapping area is not the probability of failure but the probability of failure can be calculated as follows:

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* Failure will be considered to have occurred when any of the limit states is reached.

$$\Pr(\text{Failure} \quad q < Q < q + dq) = \int_{-\infty}^{\infty} f_Q(q) \left[\int_{-\infty}^{q} f_R(r)\,dr \right] dq \qquad (2)$$

where f_Q and f_R are probability density functions (PDF) of Q and R respectively. Therefore to obtain a probability of failure one requires probability distributions of Q and R.

3. PROBABILISTIC MODELS

3.1 Definition

A probability distribution is a mathematical description of the uncertainty regarding a random quantity or a set of random quantities. A model consists of a general mathematical form and is defined for a particular sample/population by the parameter (s) of that sample population. When a probabilistic model is sought to fit a set of data, the model should closely reproduce the mean, mode, skewness and peakedness of the data. The model should preferably be defined only over the range for which the random quantity exists; for example, strength is always positive. Sometimes the underlying physical phenomena governing the characteristics may suggest the form of the model; as an example, a random quantity which is understood to be generated as a product of other quantities may be suitably represented by a lognormal model.

Unless the size of the data is large, the analyst should only insist that the model fit the data reasonably well, because there is always a finite probability that a sample will not reproduce the underlying distribution. Statistical conventions do not strictly define limits for a "reasonable" fit and the analyst will normally have to make a subjective judgement. It is also a question of subjective judgement whether or not a series of tests constitutes an appropriate simulation of a practical situation.

3.2 Data base

Data for strength, elastic and creep values for clay masonry are now examined. It has been difficult to obtain data for large samples. Although a large number of strength tests (as well as a significant number of elastic modulus tests) have been conducted over the years, hardly any two researchers have used similar types of bricks. We have chosen only those research reports having the largest number of results which results, in our opinion, were obtained from similar tests on similar materials.

3.3 Compressive strength of brickwork prisms

Figure 2a shows a histogram for the compressive strength test data for brickwork prisms (Powell and Hodgkinson [5]). Powell and Hodgkinson [5] tested 2-wide, 8-high brick prisms containing 1:¼:3 mortar. For the three types of bricks included in these results the brick type did not have any significant effect on strength of masonry. The sample mean and standard deviation are used as estimates of the*population mean and standard devia-

tion. The normal PDF n(x;25.45,4.876) is superimposed over the histogram and shows a reasonable fit (Fig. 2a). The cumulative distribution function (CDF) of the normal distribution also compares favourably with the cumulative distribution of the data (Fig. 2b).

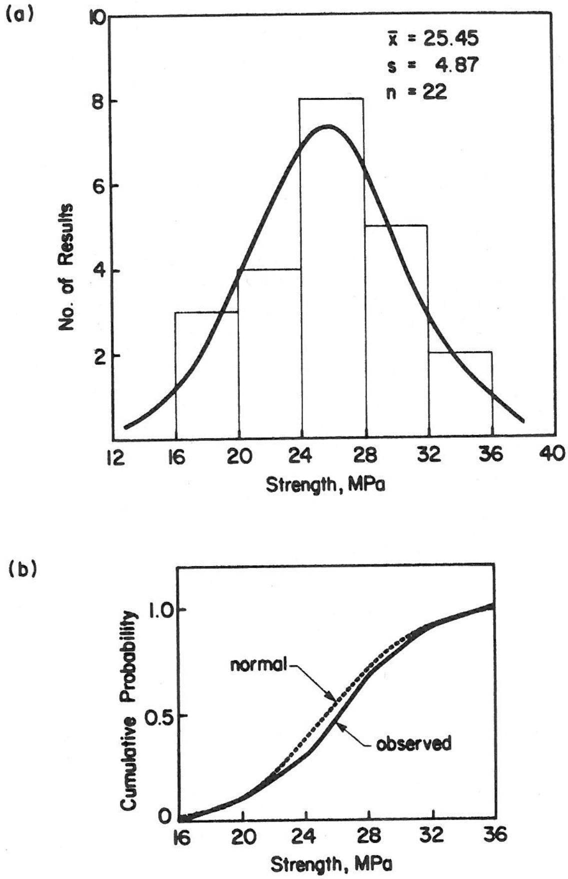

Fig. 2 Histogram and Ogive for the compressive strength of brickwork

The χ^2 (chi-square) goodness-of-fit test was used to confirm the suitability of the model, which was supported by this test (at the 5% level of significance).

3.4 Elastic modulus of brickwork

Data for the brickwork elastic modulus were taken from Lenczner [6], Lenczner et al [7], and Lenczner and Salahuddin [8]. All the prisms tested were made with 1:¼:3 mortar. The measured value of the elastic modulus was not significantly affected by the brick type. A normal

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* The numbers refer to the mean and standard deviation respectively.

distribution was tried and the agreement between model and data was good. The χ^2 goodness-of-fit test confirmed that the normal model n(x;16720,6191 MPa) is acceptable. This model gives a 0.35% probability of obtaining negative values of the modulus, but is simpler to use than the truncated version, which is

$$f_x(x) = 1.0035\ [n(x;16720,6191)],\quad x \geqslant 0$$
$$= 0, \qquad x < 0 \tag{3}$$

3.5 Creep of brickwork

It was found that the gamma distribution g(x;0.0505,1.47 μ/MPa) was suitable for representing the data for the specific creep (creep per unit stress) of brickwork. Data were obtained from the reports of Lenczner [6] and Lenczner et al [7], who used four brick types and 1:¼:3 mortar to make hollow piers which were then loaded at ages of 14 and 28 days. Neither the brick type nor the change in age of loading had any significant effects on the recorded creep values.

All the distribution models are given in Table 1, where models for concrete masonry (Ameny [20]) are also shown for comparison.

TABLE 1

PROBABILITY DISTRIBUTION MODELS FOR MASONRY

	Strength (MPa)	E (MPa)	Creep/Stress (μ/MPa)
Concrete Masonry	$n_F(f;9.18,1.133)$	$n_E(e;6030,751)$	$n_C(c;156,17.9)$*
Clay Masonry	$n_F(f;24.45,4.867)$	$n_E(e;16720,6191)$	$g_C(c;0.0505,1.47)$†

* After 110 days under load
† "Ultimate" creep

4. BAYESIAN STATISTICAL APPROACH

4.1 Introduction

Because of the limited amount of data at present available on various masonry properties, the masonry designer or researcher may be faced with considerable uncertainty as to what values to adopt for his design. It was shown in section 2 that variability in masonry properties can be very high. CSA Standard CAN3-S304-77M provides for the variability in the strength of test prisms by stipulating that the mean test value be

multiplied by a factor of 1-1.5V, where V is the coefficient of variation. In coming up with such a factor an assumption had to be made regarding the probability distribution model for strength. Therefore some sort of prior knowledge about the random quantity was assumed.

The authors of this paper wish to suggest that the use of a Bayesian statistical approach leads to a direct and logical incorporation of prior knowledge or assumptions. The approach is based on the familiar Bayes' theorem:

$$p(e|f) = \frac{p(f|e)p(e)}{p(f)}$$

where e may be the elastic modulus and f the compressive strength of masonry. p(e) is a prior distribution and should correctly reflect a prior state of knowledge, that is, it should be only as sharp as the prior knowledge justifies. p(f|e) is called the likelihood and should be obtained by a regression analysis of f on e and not e on f (Jordaan [9]).

4.2 Linear regression - classical approach

Linear regression models are very common in many engineering branches. As an example, it may be noted that researchers have frequently tried to relate the elastic modulus to the strength of the material, and all formulations for masonry have been of the linear from (Khalil [10]):

$$E = \alpha_1 + \beta_1 F \tag{5}$$

where E = elastic modulus
F = compressive strength of masonry
α_1 and β_1 are usually denoted as constants but are really random quantities. The above equation is obtained from linear regression analysis as

$$\mathcal{E}(E|F = f) = \alpha_1 + \beta_1 f \tag{6}$$

where $\mathcal{E}$ denotes expected value. From the available data point estimates $\hat{\alpha}_1$ and $\hat{\beta}_1$ are obtained for α_1 and β_1.

For an illustrative example a set of 39 results were taken from Plowman [11], Lenczner [12], Lenczner et al [7], Powell and Hodgkinson [5], Ameny [13], and Khalil [10]. The following results are obtained:

$$\hat{\alpha}_1 = 1396 \text{ MPa}; \; \hat{\beta}_1 = 688 \text{ MPa}$$

$$\mathcal{E}(E|F = f) = 1396 + 688 f \tag{7}$$

The regression line is shown in Fig. 3. It is seen that the equation E = 1000 f'_m will almost always over estimate the elastic modulus. On the other hand the line E = 800 f'_m lies within the 95% confidence limits and seems to be a much better estimation.

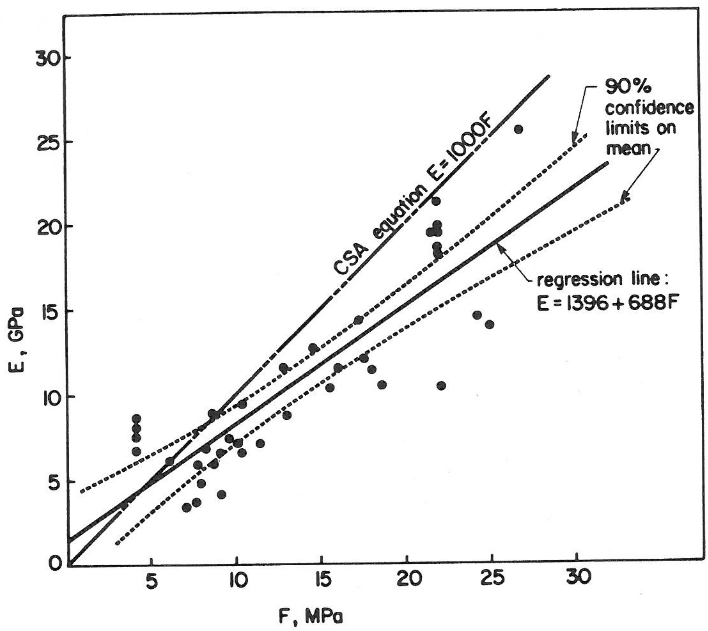

Fig. 3 Regression line for the relationship between elastic modulus and strength (classical approach)

4.3 Regression analysis - Bayesian approach

In the regression analysis using Bayesian approach, it is possible to incorporate a prior belief about what the distributions of α, β and E should be.

The analysis requires knowledge of the likelihood function $p(f|e)$ and it is assumed that the regression of f on e is linear. For simplicity assume the regression to be homoscedastic (Lindley [4]), that is the variance $V(f|e)$ is constant ϕ for all values of e. It can be shown (Lindley [14], Novick and Jackson [15], Maddock [16]) that if (i) a normal model can be assumed for the strength distribution, (ii) independent indifference priors can be used for α, β and ϕ, (iii) n, the number of data points is large, (iv) a normal prior $n(e;\mu_2,\sqrt{\sigma_2})$ is taken for the elastic modulus, then the following (approximate) normal posterior density is obtained:

$$p(e|f) = n(e;\mu_o, \sqrt{\phi_o}) \tag{8}$$

$$\mu_o = \left\{\frac{\hat{\beta}_1(f-\hat{\alpha}_1)}{\phi_1} + \frac{\mu_2}{\phi_2}\right\} \Big/ \left\{\frac{\hat{\beta}_1^2}{\phi_1} + \frac{1}{\phi_2}\right\} \tag{9}$$

$$\frac{1}{\phi_o} = \frac{\hat{\beta}_1^2}{\phi_1} + \frac{1}{\phi_2} \tag{10}$$

$$\hat{\beta}_1 = \sum_{i=1}^{n} (e_i - \bar{e})(f_i - \bar{f}) \Big/ \sum_{i=1}^{n} (e_i - \bar{e})^2 \tag{11}$$

$$\hat{\alpha}_1 = \bar{f} - \hat{\beta}_1 \bar{e} \tag{12}$$

$$\phi_1 = \left[1 + \frac{1}{n} + \frac{(e_{n+1} - \bar{e})^2}{\sum_{i=1}^{n} (e_i - \bar{e})^2} \right] \frac{s^2}{n-2} \tag{13}$$

$$s^2 = \sum_{i=1}^{n} (f_1 - \bar{f})^2 - \frac{\sum_{i=1}^{n} (e_i - \bar{e})^2 (f_i - \bar{f})^2}{\Sigma (e_i - \bar{e})^2} \tag{14}$$

n is the number of data pairs, e_i and f_i are the individual data and $\bar{e}$ and $\bar{f}$ are the corresponding data means. e_{n+1} is the e value used to predict f_{n+1}.

The data used in the example of the last section is now reanalyzed. It should be emphasized that the normality of E depends on the normality of $p(f|e)$; the latter assumes a large data set, thus making the dependence of ϕ on e negligible. On this basis, we obtain $\phi \simeq 13.34$. Two prior distributions were used, (1) n(e;,19962,2251 MPa) and (2) n(e;,19962,9000 MPa). The posterior distributions are approximately normal, that is

$$n(e;\ 13335 + 282.4f,\ 1883 \text{ MPa}) \tag{15}$$

and

$$n(e;\ 685.4 + 821.7f,\ 3212 \text{ MPa}) \tag{16}$$

corresponding to the two priors respectively. The two lines and 95% credibility intervals are shown in Fig. 4.

It is clear that the incorporation of a sharp prior -(1) above - results in a regression line that is very different from the line of Fig. 3 whereas a more diffuse prior -(2) above - results in a line much closer to that of Fig. 3.

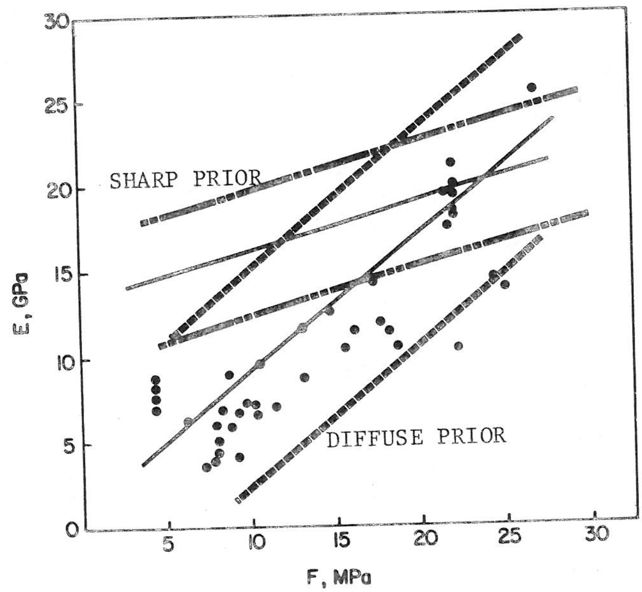

Fig. 4 Regression line from Bayesian analysis, showing the marked effect of using a prior distribution

5. Example: Probability of excessive deflections

There are times when deflection control for a masonry beam is a more severe limitation than stress considerations (Schneider and Dickey [18]). On the basis of a set of aesthetic and mechanical desiderata the codes specify maximum allowable deflections, usually as the ratio w/ℓ, where w and ℓ are deflection and span, respectively.

It is possible to assume that the deviations of ℓ from the specified value are negligibly small. A similar assumption can not in general be made for the moment of inertia I of a section, but for ease of mathematical manipulation, the deviations of I from the specified value will be neglected.

The deflection ratio for a simply supported beam under uniformly distributed load is

$$\frac{w}{\ell} = \frac{5}{384} \frac{\ell^3 Q}{EI} \tag{17}$$

where Q is a uniformly distributed (live) load and E is the elastic modulus. Equation (17) can be written as

$$\Delta = CQK \tag{18}$$

where $\Delta = w/\ell$, $C = 5\ell^3/384I$ and $K = \dfrac{1}{E}$

Q is random quantity and will be assumed to have a lognormal probability distribution, as suggested by Corotis and Doshi [19]. K is also a random quantity and on the basis of data given by Ameny [13] K may be deemed to have a lognormal distribution with a mean of 0.169/MPa and standard deviation of 0.023/MPa.

Consider a simply supported masonry beam with a span of 6 m and supporting a selfweight and specified live load of 12.5 kN/m. f'_m = 10 MPa and the dimensions are b = 250 mm, h = 900 mm, d = 825 mm. Steel reinforcement properties are: f_y = 300 MPa, $f_{s(allowable)}$ = 150 MPa, E_s = 200,000 MPa, E_s/E_m = 20. What is the probability of the deflection ratio exceeding the stipulated maximum of $\ell/300$?

If C is a constant and Q and K are lognormally distributed, then Δ is also lognormally distributed. It is easy to show that

$$m_\Delta = 1.301 \times 10^{-3}, \quad m_{\ell n\Delta} = -7.056, \quad \sigma^2_\Delta = 9.114 \times 10^{-8},$$

$$\sigma^2_{\ell n\Delta} = 0.1047$$

If $x = \ell n\Delta$, the probability of excessive deflection is

$$p_f = \int_{\ell n\Delta'}^{\infty} \frac{1}{\sigma_x \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-m_x}{\sigma_x}\right)^2\right] dx \tag{19}$$

where $\Delta' = \ell/300 = 0.02$

It is found that $p_f = 1.27 \times 10^{-21}$

The above procedure can be used to obtain other probabilities. It is not always possible to obtain closed form expressions for the probability. Numerical analysis techniques can then be utilized to evaluate the probability.

6. CONCLUDING REMARKS

It has been shown that masonry characteristics are random quantities with high coefficients of variation, and can be suitably represented by well-known probability functions. There is an urgent need for the generation and collection of all types of masonry data required in structural engineering decision making. The limited quantity of masonry data at present available requires very careful analysis and much personal judgement. Bayesian analysis is a proper way of incorporating prior knowledge in the analysis.

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