A NUMERICAL MODEL FOR MASONRY

F. SAWKO Professor
Department of Civil Engineering, University of Liverpool, Liverpool, U.K.

M. A. ROUF Assistant Professor
Department of Civil Engineering, University of Dakha, Bangladesh

ABSTRACT The paper describes a numerical model developed for the structural analysis of masonry elements subjected to axial force, bending moment and shear. The model assumes a parabolic stress-strain relationship in compression and neglects tensile strength. Formulae are derived for axial and bending stiffness properties for elements subjected to a condition of compressive forces and bending moments and for shear stress distribution across a masonry element. The numerical model is applied to the analysis of eccentrically loaded walls and piers and masonry arches.

1. INTRODUCTION

Current design philosophy for masonry construction is based on the concept of limit states (1). It is therefore necessary for the designer to be able to predict the ultimate limit state (collapse load) for a structure and its constitutive elements.

Linear elastic analysis cannot be utilised for this purpose because of non-linearity of response close to the ultimate limit state. This difficulty is not confined to masonry, and structures in steelwork and concrete have posed similar problems. These have been overcome by defining approximate properties of materials on which a suitable analytical model could be based. Thus for steelwork a rigid-plastic model has led to the development of the well known plastic (or collapse) analysis, which was further refined by using an elasto-plastic model (with strain hardening, as appropriate) and the whole procedure computerised. Similar developments have taken place in reinforced concrete. The Analysis of Masonry Structures has lagged behind, and it is the Authors' contention that the main reason for this was lack of the basic definition of material properties. The paper suggests a numerical model which the Authors have developed and applied to the ultimate limit state analysis of masonry structures, and which they consider suitable for further refinement and application to a variety of structural forms in masonry.

2. BASIC ASSUMPTIONS

The assumptions employed in the development of the numerical model are based on the observed behaviour of masonry and can be summarised as follows:

a. Non-linear stress-strain relationship in compression.

b. Negligible tensile strength.

c. Linear strain distribution.

There is empirical evidence (2) that stress-strain relationship for masonry prisms, that is brick and mortar elements, is non-linear. Figure 1 reproduced from the report shows for four different types of brickwork a flattening of the stress-strain curve up to a peak value followed by a falling branch. This behaviour can be approximated by a parabola (Fig. 2) expressed in dimensionless form as:

$$\frac{\sigma}{\sigma_m} = 2\frac{\varepsilon}{\varepsilon_m} - \left(\frac{\varepsilon}{\varepsilon_m}\right)^2 \qquad (1)$$

in which $\frac{\sigma}{\sigma_m}$ is the stress ratio, that is ratio of actual to maximum stress and $\frac{\varepsilon}{\varepsilon_m}$ the corresponding strain ratio. Thus the stress-strain relationship of any masonry element in comparison can be defined by specifying numerical values of σ_m and ε_m.

The inclusion of the falling branch is important and, in the Authors' opinion, vital for the successful development of the numerical model. Specimens shown in Fig. 1 indicate a substantial strain past ε_m. Theoretical studies carried out by the Authors suggest that the curve could be curtailed at 1.5 ε_m without any loss of generality.

Masonry does possess some tensile strength which is important in some structural forms, for example masonry panels. For elements subjected to eccentric compresive loading, it contributes little to the overall strength, and has been neglected in the present study. Its inclusion in a more refined model should not, however, present any real difficulty.

The assumption of linear strain distribution is adopted universally for most non-linear theories and is not thought to be unrealistic for masonry construction.

3. STIFFNESS PROPERTIES

The definition of stiffness properties for axial force and bending moment are a pre-requisite for any structural analysis. For an elastic element in compression, axial stiffness is obtained by dividing the load increment by the consequential contraction, giving:

$$EA = \frac{\delta p}{\delta \varepsilon}$$

The stiffness is constant along the element and defined in terms of E the Youngs' modulus of element material and A its cross sectional area.

If non-linear behaviour is assumed, the stiffness becomes a function of not only E or A, but also of the state of strain (and hence stress). Since stress can vary along the length of an element, it is no longer possible to develop explicit expressions for overall stiffness of an element, but it is possible to derive expressions for stiffness corresponding to a particular strain distribution at any section. Furthermore, for non-linear stress-strain relationship, the axial and bending stiffnesses interact and can be expressed (3) as:

$$\begin{bmatrix} \delta_p \\ \delta_m \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} \delta\varepsilon \\ \delta\theta \end{bmatrix} \qquad (2)$$

For an uncracked section, shown in Fig. 3, the stiffness matrix becomes:

$$[S] \quad \left[\begin{array}{c|c} E_0A_0 \{1 - \frac{\varepsilon_1}{\varepsilon_m} \frac{(1+R)}{2}\} & \frac{-E_0I_0}{T} \frac{\varepsilon_1}{\varepsilon_m} (1-R) \\ \hline -E_0A_0 \frac{T}{12} \frac{\varepsilon_1}{\varepsilon_m} (1-R) & E_0I_0 \{1 - \frac{\varepsilon_1}{\varepsilon_m} \frac{(1+R)}{2}\} \end{array}\right] \tag{3}$$

These terms can be expressed in graphical form shown in Fig. 4. It will be observed that for R=1, that is uniform strain across the element, the axial stiffness term for any strain level is expressed as the tangent to the curve, and changes from the maximum value of E_0A_0 for zero strain level, to zero at ε_m and becomes negative for strains exceeding ε_m. Stiffness properties of cracked elements are also derived and expressed in numerical and graphical form in Ref. 3. Stiffness properties thus derived permit the formulation of general purpose computer based program for the structural analysis of masonry structures comprised of beam and column elements.

4. SHEAR STRESS DISTRIBUTION

For linear elasticity, it is possible to derive the following well known expression:

$$s = \frac{VA\bar{y}}{IB} \quad \text{when} \tag{4}$$

s = shear stress at fibre y from centroidal axis
V = shear force acting on the section
$A\bar{y}$ = first moment of area of element about centroidal axis
B = width of member
I = second moment of area of the section about centroidal axis

For the particular case of a rectangular section, equation (4) can be expressed as:

$$s_a = \frac{V}{BT} \left(\frac{1}{4} - \frac{a^2}{T^2}\right) \quad \text{when} \tag{5}$$

T = thickness (depth) of section
a = distance of fibre from centroidal plane.

Equation 5 gives the well known parabolic distribution of shear stresses with depth for a rectangular section, with zero shear stresses at extreme fibres and maximum stress at the centroidal axis. It is evident that, in linear elasticity, shear stress distribution is a function of the shear force V but independent of moment M acting at the section.

In non-linear elasticity, shear stress depends, additionally, on the state of strain at the section, which in turn is a function of the bending moment acting at the section. Thus, two sections each with the same shear force but different bending moments would exhibit different shear stress distributions.

With reference to Figure 5, shear stress distribution for non-linear behaviour is determined by a procedure similar to the one adopted in linear elasticity, that is by horizontal equilibrium of a small block uvmn. The algebraic expressions derived (4) are rather complex and are not reproduced here, but the distribution of shear stresses in a rectangular section is plotted in Fig. 6, for the

particular case of $\varepsilon_2 = 0$. It will be observed that in the absence of any strain on the section, i.e. $\varepsilon_1 = 0$, the elastic parabolic stress distribution is reproduced, (Fig. 6a), but stresses depart significantly from this solution with increasing ε_1. A similar situation exists for surface applied shear shown in Fig. 6b.

5. EXAMPLES OF LIMIT STATE ANALYSIS

(a) Slender Walls and Piers

The numerical model based on the parabolic stress-strain distribution has been applied to the analysis of slender walls with eccentric loading and different end conditions (5-7). The problem is a complex one, since the lateral deformation of the loaded wall modifies the initial eccentricity of any load at any section. It has been possible to formulate a successive correction procedure by which, for a particular load application, a convergent solution is obtained. Load is increased incrementally until collapse.

An example of such analysis is taken from the Authors' contribution (9) to the discussion of a paper by J. Sutherland (8). It is evident from Fig. 7 that the method is capable of tracing the load deflection characteristics of the wall up to its ultimate limit state, for different material properties. At maximum load, the load deflection curve becomes horizontal and the lateral stiffness of the wall is reduced to zero. At this point, the lateral deflections are small compared with the thickness of the element (plotted to scale in Fig. 7b) but would increase rapidly if the maximum load was maintained. It will further be observed, that at maximum load P the strain ratios are low and the wall exhibits a classical buckling behaviour. This predicted behaviour reproduces the behaviour observed under laboratory conditions, when it was possible to remove the load near its maximum (indicated by the horizontal portion of the load-deflection curve) and to leave the specimen undamaged.

(b) Masonry Arches

Considerable interest has recently been demonstrated in the structure behaviour of masonry arches. Reasons for this interest are two-fold: the need to assess the strength of existing structures (which in the U.K. number many tens of thousands) and the renewed interest of architects and engineers in the arch as a modern structural form.

Methods for structural analysis of arches fall into three categories:

(a) Elastic method developed by Pippard (10).

(b) Collapse method developed by Heyman (11).

(c) Finite element method developed by Sawko and Towler (12, 13) and others (14, 15).

Elastic method gives information about the behaviour of an arch up to the formation of the first crack and is of no value in predicting the load and mode of collapse. Collapse method assumes the formation of "plastic hinges" with regions between them remaining rigid and undeformed. There is a further complication in the assumption of a uniform 'vertical' thickness of the arch rib, and of infinite compressive strength of masonry. The various finite element approaches are the only methods capable of tracing the progression of cracking up to failure.

At the Department of Civil Engineering at Liverpool University three model arches shown in Fig. 8 were tested to destruction. Arches Nos. 1 and 2 failed by an instability mode similar to that predicted by the collapse method, but the third failed in shear as shown in Fig. 9.

A numerical model developed by Sawko and Towler (12) was used to assess their theoretical behaviour. Results of these analyses are summarised in Table 1. The method did not include a shear failure mechanism and is shown to grossly overestimate the collapse load for arch No. 3.

The early model has now been refined (14) by incorporating an improved convergence criteria. Results of analysis of arches 2 and 3 at ultimate limit state are shown in Fig. 10. More important, the shear failure mechanism discussed in section 4 has also been incorporated. It will be observed that the shear failure mode for arch 3 can now be successfully modelled.

6. CONCLUSIONS

The numerical model developed by the Authors has been shown to be capable of analysing different structural forms up to their ultimate limit state. It is capable of further refinement and extension into other structural forms such as panels. The Authors feel that its universal adoption would encourage further development of automated computer based methods which would take structural masonry from its middle ages image into the modern computer era.

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TABLE 1 (FAILURE LOAD OF ARCHES (in kN)

		Previous Nonlinear Analysis [13]	Experimental	Current Nonlinear Analysis	
				Without Shear Failure Criterion	With Shear Failure Criterion
ARCH 1 (2 Ring)	Crown load	100.0	82.5[13]	90.0	83.0
ARCH 2 (3 Ring)	Q.P. load	102.0	117.0[13]	85.0	83.0
ARCH 3 (3 Ring)	Crown load	950.0	380.0[13]	1410.0	400.0

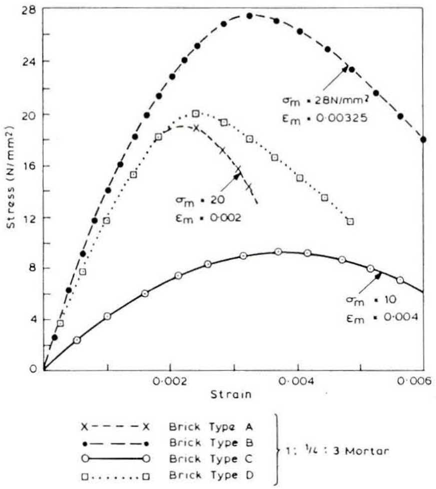
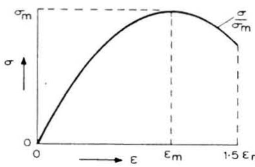


FIGURE 1 EMPIRICAL STRESS - STRAIN RELATIONSHIP FOR BRICKWORK.[2]



$$\frac{\sigma}{\sigma_m} = 2\frac{\varepsilon}{\varepsilon_m} - \left(\frac{\varepsilon}{\varepsilon_m}\right)^2$$

FIGURE 2 IDEALISED STRESS - STRAIN RELATIONSHIP FOR MASONRY.

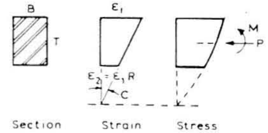


FIGURE 3. STRAIN AND STRESS DISTRIBUTION IN UNCRACKED SECTION.

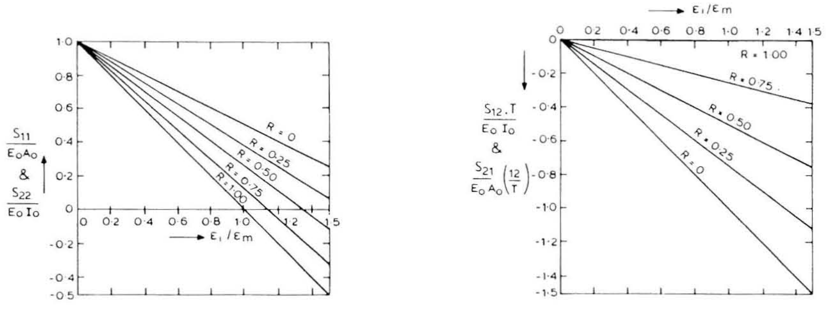
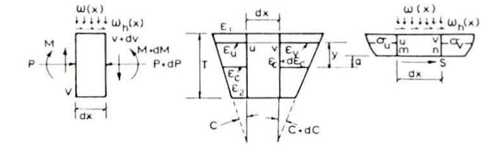


FIGURE 4. STIFFNESS COEFFICIENTS FOR UNCRACKED SECTION.



(a) Elemental Length of a member (b) Strain Distribution (c) Stresses on a slice of the element

FIGURE 5 FORCES, STRAINS AND STRESS IN AN ELEMENTAL LENGTH OF A MEMBER

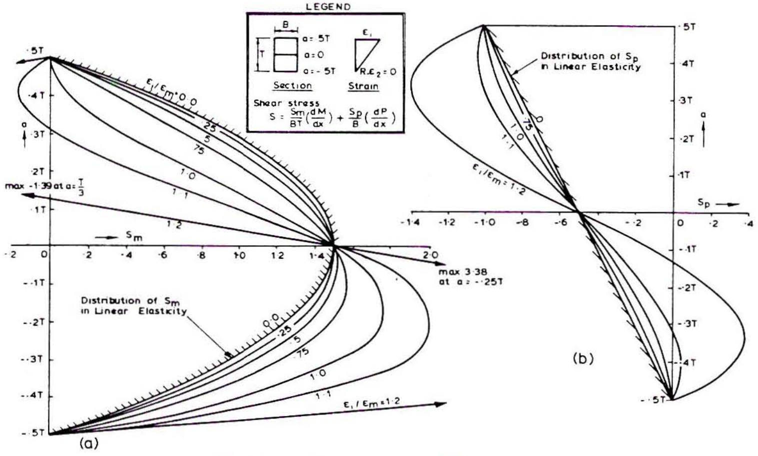
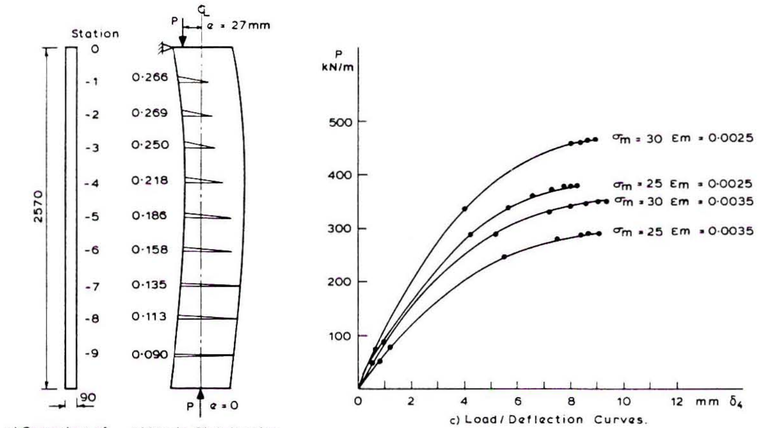


FIGURE 6. DISTRIBUTION OF SHEAR STRESS TERMS



a) Geometry of Cross Section.

b) Strain Distribution and Deflection σ_m = 30N/mm² εm = 0·0035

c) Load / Deflection Curves.

FIG. 7. LOAD DEFLECTION CURVES AND STRAIN DISTRIBUTION OF ECCENTRICALLY LOADED SLENDER WALL. (H/t = 28·6, e_t = 0·3 , e_b = 0)

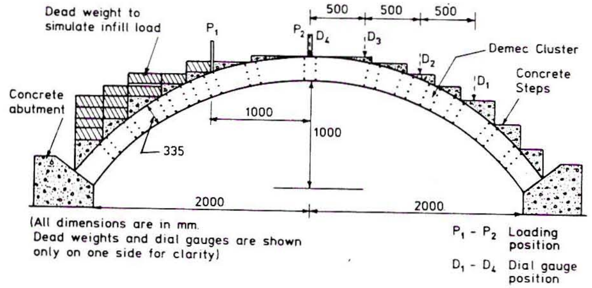


FIG. 8. INSTRUMENTATION AND LOADING CONDITION FOR BRICKWORK ARCH

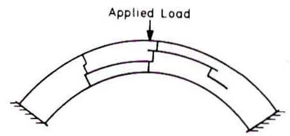


FIG. 9. SHEAR FAILURE IN BRICKWORK ARCH

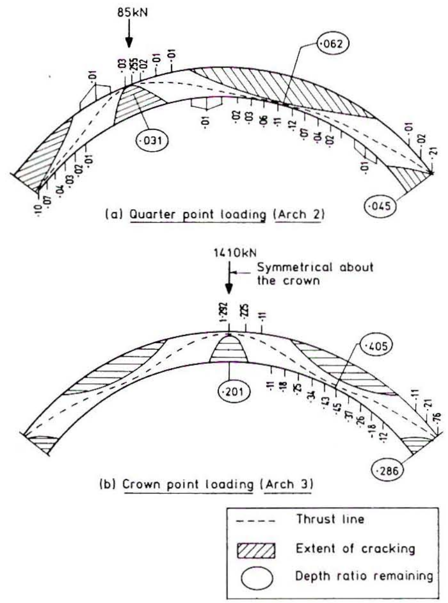


FIG. 10. STRAIN DISTRIBUTION AT ULTIMATE LIMIT STATE