A CONSTITUTIVE MODEL FOR STRUCTURAL MASONRY

Vassili V.Toropov¹, Stephen W.Garrity² and Genadiy V.Turovtsev³

1. ABSTRACT

A new approach to describe the average mechanical properties of structural masonry is presented. A macroscopic element of structural masonry is considered as a homogeneous and anisotropic medium with stress-state type dependent properties. A general formulation for such materials is provided and applied to model the non-linear elastic-brittle behaviour of masonry. The methodology followed is based on a systematic employment of Lode's angle in the constitutive relationships. A general failure criterion for materials with stress-state type dependent properties is also incorporated into the model. The constitutive relationships are presented in a tensor linearised form that is suitable for use with numerical methods of analysis.

2. INTRODUCTION

Intensive investigations made over recent years have led to an improved understanding of the constitutive behaviour of masonry under various conditions of loading. The results of numerous load tests clearly indicate that one of the most important features of unreinforced masonry behaviour is the dependence of the material properties on the type

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¹ Reader and ² Senior Lecturer, Department of Civil & Environmental Engineering, University of Bradford, Bradford, West Yorkshire, England. BD7 1DP.

³ Professor, Department of Mathematical Methods & Information Technology, Zaporozhye Institute of Technology, 16a Kiyashko Street, Zaporozhye, 330015, Ukraine.
of the stress state, for example whether the material is subjected to tension or compression. This dependence has been observed in masonry in both the elastic stress range and at failure [1]. Research has shown that the Young’s modulus of the mortar joints in compression can be in the order of two to three times higher than that in tension [2] and, when considering the flexural behaviour of masonry, different stiffnesses in tension and compression have been used [3,4].

Various models describing the behaviour of masonry have been proposed; most of these are of the phenomenological type. As far as the authors are aware, no systematic approaches to model the stress-state type dependent properties of masonry have been developed. Such an approach is warranted as the stress-state type dependence has a significant influence on the elastic behaviour and the predicted value of the failure load. In this paper, an elastic-brittle model is developed for masonry based on the non-linear theory of stress-state type dependent solids [5]. One of the principal features of the model is to consider masonry as a stress-state type dependent medium rather than a non pressure dependent medium.

3. GENERAL STRUCTURE OF THE MODEL

The model is based on the assumption that masonry exhibits an initially isotropic continuum response and three distinct stages of behaviour, namely: linear elasticity, non-linear behaviour of the damaged material and failure. All these stages can be associated with the corresponding damage mechanism of a homogeneous material with average properties. No attempt has been made to study the micromechanism of damage or single crack propagation or to model the time dependent behaviour of masonry. It should be remembered, however, that masonry is a non-homogeneous anisotropic material and it is therefore impossible to attribute local damage of a real material to a single cause of cracking. Although the anisotropic behaviour of masonry has not been accounted for in the constitutive model described, it is the subject of future development.

3.1 Failure criteria

It is assumed that, as in plasticity theory, an initial cracking surface \( F(\Sigma) = 0 \) and failure surface \( \Phi(\Sigma) = 0 \) define the upper bound and lower bound, respectively, for an effective stress \( \Sigma \) in the stress space. If the stresses lie within the limiting surface \( F(\Sigma) = 0 \), the material is assumed to be in a linear elastic state. When the stress locates at the upper bound \( \Phi(\Sigma) = 0 \), failure of the material occurs in the form of either cracking or crushing depending on the state of stress. After the ultimate strength is reached, the masonry is unable to support any load. If the masonry is stressed beyond the limiting surface \( F(\Sigma) = 0 \), it is assumed that damage occurs and microcracks develop and propagate in a stable manner.

It is evident from previous research that the elastic properties of masonry in tension differ from those in compression. In this paper, masonry is considered to exhibit a more general continuous dependence of its elastic properties on the stress-state type: this dependence is treated as a major phenomenon of masonry behaviour.
It is well established in the literature [6,7] that the trace of the failure function $\Phi(\Sigma)$ on the deviatoric stress plane seems to be non-circular or Lode angle dependent. Aiming to study the influence of the stress-state type on the masonry material behaviour, it is assumed that the initial cracking criterion $F(\Sigma) = 0$ and failure criterion $\Phi(\Sigma) = 0$ are expressed in terms of the octahedral shear stress and Lode angle, i.e., $\Sigma = \Sigma(\tau, \phi)$. Hence, similar to von Mises type theories, the limit states of the masonry materials are associated with the magnitude of the distortion strain energy, $W(\Sigma)$. Assume $W(\Sigma)$ to be of the form

$$W(\Sigma) = B\Sigma^2,$$

and assume the failure criterion in the form

$$\Phi(\Sigma) = W(\Sigma) - k^2 = B\Sigma^2 - k^2 = 0.$$  

If the fracture criterion $F(\Sigma) = 0$ has the same functional form as failure criterion (2), then

$$F(\Sigma) = W(\Sigma) - k^2 = C\Sigma^2 - k^2 = 0$$

The parameters in the above functions can be determined from tests, with loading sustained until the proportional limits and the limit states in tension, compression and shear have been reached. Thus, the initial cracking surface and failure surface can be defined, as shown in Figure 1. If the material is stressed beyond the initial cracking surface, it is assumed that it is still in an elastic state but that the response of the material differs under different types of loading, e.g. tension, compression and shear. At this stage, although the masonry is cracked, it is assumed that it is still capable of sustaining an applied load in a stable manner. The classical theories of elasticity are not valid to describe material behaviour in this stage and a more general theory is required.

![Figure 1. Deviatoric sections](image-url)
3.2 Stress-strain relationships

Consider two symmetric second order tensors $\sigma_{kl}$ and $\varepsilon_{kl}$ in the Cartesian co-ordinate system. As it follows from the representation theorem the general form of the relationship between these tensors is:

$$\varepsilon_{kl} = c_0 \delta_{kl} + c_1 \sigma_{kl} + c_2 \sigma_{ks} \sigma_{sl} \quad (k,l = 1,2,3) \tag{4}$$

where $\delta_{kl}$ is the unit tensor and $c_i$ are the scalar coefficients. Assuming the tensors $\sigma_{kl}$ and $\varepsilon_{kl}$ to be co-axial, (4) may be presented in a form where the coefficients $c_i$ are expressed as functions of invariants of the tensors. Following the work by Tsvelodub [8], this relationship can be represented in the form

$$\varepsilon_{kl} = 3e \frac{\partial \sigma}{\partial \sigma_{kl}} + W \left( \frac{1}{\tau} \frac{\partial \tau}{\partial \sigma_{kl}} - \tan \omega \frac{\partial \varphi}{\partial \sigma_{kl}} \right) \quad (k,l = 1,2,3) \tag{5}$$

where $\varepsilon_{kl}^0 = \varepsilon_{kl} - \delta_{kl}e$, $\sigma_{kl}^0 = \sigma_{kl} - \delta_{kl}\sigma$ are the tensor deviators; $e = \frac{1}{3} \varepsilon_{kk}$, $\sigma = \frac{1}{3} \sigma_{kk}$, $\gamma = (\frac{1}{3} \sigma_{ll}^0 \sigma_{kl}^0)^{\frac{1}{2}}$, $\tau = (\frac{1}{3} \sigma_{ll}^0 \sigma_{kl}^0)^{\frac{1}{2}}$ are the octahedral normal and shear stresses of the corresponding tensors; $\omega = \varphi - \psi$ is a phase of similitude of the tensor deviators; $\varphi$ and $\psi$ are the Lode angles, such that $\cos 3\psi = \sqrt{2} \gamma^{-3} \det \sigma_{kl}^0$ and $\cos 3\varphi = \sqrt{2} \tau^{-3} \det \varepsilon_{kl}^0$; $W = \varepsilon_{kl}^0 \sigma_{kl}^0 = 3\tau \gamma \cos \omega$ is the mixed invariant of the tensors or specific distortion energy. The similar inverse relationships $\sigma_{kl} = \sigma_{kl} (\varepsilon_{kl})$ exists.

If a function $\Phi(\sigma, \tau, \varphi)$ exists such that

$$e = \frac{1}{3} \frac{\partial \Phi}{\partial \sigma}; \quad W = \tau \frac{\partial \Phi}{\partial \tau}; \quad \tan \omega = -\frac{\partial \Phi}{\partial \tau}$$

then, (5) can be expressed in the form

$$\varepsilon_{kl} = \frac{\partial \Phi(\sigma, \tau, \varphi)}{\partial \sigma_{kl}} = \frac{\partial \Phi}{\partial \sigma} \frac{\partial \sigma}{\partial \sigma_{kl}} + \frac{\partial \Phi}{\partial \tau} \frac{\partial \tau}{\partial \sigma_{kl}} + \frac{\partial \Phi}{\partial \varphi} \frac{\partial \varphi}{\partial \sigma_{kl}} \quad (k,l = 1,2,3). \tag{6}$$

In this case, the problem of development of relationships between two coaxial tensors reduces to the definition of a single potential function $\Phi(\sigma, \tau, \varphi)$. The existence of the potential function provides a sufficient condition of the stability of the material, but since $\partial \Phi/\partial \sigma_{kl}$ contains the tensor-nonlinear term, the resulting relationships would be cumbersome to use in applications. Complexity of these relations seems to be unwarranted.

Consider a more general way of stress-strain relations development, based on relationship (5). In general, a single potential function for the tensor $\varepsilon_{kl}$ does not exist and the assumption of existence of the potential function limits the class of possible relationships between tensors $\sigma_{kl}$ and $\varepsilon_{kl}$. From (5), in order to make the relationship
between two coaxial symmetric second order tensors completely definite, in general, it is necessary to specify three functions \( e(\sigma, \tau, \phi) \), \( W(\sigma, \tau, \phi) \) and \( \omega(\sigma, \tau, \phi) \) which are related to the physical nature of the medium under consideration.

Due to the phenomenological nature of the approach followed here, the most general form (5) can be simplified by means of valid assumptions, which are dependent on the nature of the material that falls within the model. First it is assumed, following the classical theory of elasticity, that the volumetric response for the medium under consideration, is defined only by hydrostatic pressure, i.e., \( e(\sigma, \tau, \phi) = e(\sigma) \) and function \( z(\sigma) \) is integrable. Hence, a potential function \( \Phi_1(\sigma) \) exists such that:

\[
e(\sigma) = \frac{1}{3} \frac{d\Phi_1(\sigma)}{d\sigma}.
\]

The next simplification follows from the assumption that under loading of the material the phase of similitude of deviators depends solely on the angle \( \phi \), i.e. \( \omega(\sigma, \tau, \phi) = \omega(\phi) \). Or, the invariant \( \psi \) of the tensor \( \varepsilon_{kl} \) is a function of the corresponding invariant \( \phi \) of the tensor \( \sigma_{kl} \) that includes the classical assumption \( \psi = \phi \), i.e., \( \omega(\sigma, \tau, \phi) = 0 \).

Following the above assumptions, as suggested by Tsvelodub [9], relationship (5) is reduced to

\[
\varepsilon_{kl} = \frac{1}{3} \frac{\partial \Phi_1(\sigma)}{\partial \sigma_{kl}} + \frac{W(\Sigma)}{\Sigma_1} \frac{\partial \Sigma_1}{\partial \sigma_{kl}}; \quad \Sigma_1 = f_1(\phi)\tau; \quad f_1(\phi) = \exp(-\int_{\phi}^{\phi_0} \tan \omega d\phi). \tag{7}
\]

Then, there is a separation between the mean response and the deviatoric or shear response of the material, exactly as for the linear elastic material.

In respect of the specific distortion energy function \( W \), it is assumed that it is not dependent the hydrostatic pressure \( \sigma \), i.e. \( W = W(\tau, \phi) = W(\Sigma) \). Otherwise, the hydrostatic pressure will affect plastic yielding, if plastic yielding of an incompressible medium is considered as a limiting state of the elastic one.

In general \( \Sigma_1 \neq \Sigma \), since, as it follows from (7), the surface \( \Sigma_1 = \text{const.} \) defines the direction of the vector \( \varepsilon_{kl} \) while the surface \( \Sigma = \text{const.} \) defines the equal intensity distortion processes if the distortion energy is taken as the measure of intensity of the process. Hence it may be assumed that \( \Sigma_1 = B\tau \), i.e., \( f_1(\phi) = B \), and remains the previous form of effective stress \( \Sigma \). Then, the tensor-linear relationships (non-associated rule) follow from (7)

\[
\varepsilon_{kl} = \frac{1}{3} \frac{\partial \Phi_1(\sigma)}{\partial \sigma_{kl}} + \frac{W(\Sigma)}{3\tau^2} \sigma_{kl}^0, \quad k, l = 1, 2, 3 \tag{8}
\]

If the concept of the associated flow rule was applied, the strain \( \varepsilon_{kl} \) would be normal to the surface \( W(\Sigma) = 0 \). Such an assumption, however, is not justified since growth of the fracture strain is due to the formation of new microcracks or the growth of existing
fissures which are generally related to changed rheological state. Thus, in order to develop a stress-strain relationship by this approach it is necessary to define two material functions: $\Phi(\sigma)$ - the dilatation energy and $W(\Sigma)$ - the distortion energy. In [5] it was shown that the tensor linear form (8) of the general relationship (5) can be used to describe the behaviour of the solids with complex properties, when non-associativeness of the constitutive rule is acceptable. The relationships (8) can be treated as a reasonably good approximation for the tensor non-linear relationship (6). This relationship takes into account the dependence of the material properties on Lode’s angle and can be easily used in applications.

It should, however, be remembered that small hysteretic phenomena may occur during unloading and reloading of the material. Although the normality conditions for relationships (8) may be violated for some loading paths, as shown by Tsvelodub [8], Drucker’s stability postulate is fulfilled for these relationships under some restrictions of the material parameters. The uniqueness of the solution can be deduced from these conditions. When considering cyclic loading, however, it is necessary to use more complex relationships (6).

4. DESCRIPTION OF MASONRY BEHAVIOUR WITHIN THE MODEL

When hysteretic phenomena occurring during cyclic deformation are of less importance, but the requirement is to study the limit state of the material as a result of fracture or quasi-static loading, the relationships discussed in general terms in the previous sections may be used as a reasonable approximation for the general ones. These relationships take into account the dependence of the material properties on Lode’s angle and can easily be used in applications. Consider the brittle-elastic state of masonry and assume that the stress-state type affects the limiting stages and stress energy function.

4.1 Effective stress and stress-state type function

As assumed earlier, the initial cracking surface $F(\Sigma) = 0$ and the failure surface $\Phi(\Sigma) = 0$ are both dependent on the effective stress, $\Sigma$. The simplest form of the effective stress is

$$\Sigma = f(\cos 3\phi) \tau,$$

where $f(\cos 3\phi)$ is an experimentally defined material function. This may be considered as a modification of the von Mises theory accounting for the stress-state type dependence effect and compatibility with the aforementioned assumptions.

For isotropic behaviour, the medium function $f(\cos 3\phi)$ must satisfy some well known symmetry conditions [8]. The convexity of the surface $W(\Sigma) = \text{const.}$ imposes an additional condition on the function $f(\cos 3\phi)$. Taking $\tau$ and $\phi$ as polar co-ordinates in the octahedral plane the convexity condition can be expressed as

$$\tau^2 + 2(\tau^{'})^2 - \tau\tau^{''} \leq 0,$$
where the equation of the limiting surface $W(\Sigma) = \text{const.}$ is rewritten in the form $\tau = \tau(\varphi)$.

Consider two general approaches to evaluate the material function $f(\cos 3\varphi)$. In the first case consider any standard test program of proportional loading, where the limit states are reached at different stress-state types. As a minimum requirement, a set of tests should include uniaxial tension ($\cos 3\varphi = 1$) and uniaxial compression ($\cos 3\varphi = -1$) loading. Since, under proportional loading, the parameter $\cos 3\varphi$ remains constant, we can determine, using condition (4), a set of $f(\cos 3\varphi)$ values corresponding to the values of the stress-state type parameter. Using a valid approximation and taking into account the requirements of symmetry and the convexity of the limiting surface $\Phi(\Sigma) = 0$, we can obtain the material function $f(\cos 3\varphi)$.

An alternative approach to evaluate the material function $f(\cos 3\varphi)$, more suitable in applications, is based on known requirements of symmetry of the limit surface. Since, for an isotropic medium with different properties in tension and compression, the failure surface in the deviatoric plane has $2\pi/3$ period and $\pi/3$ symmetry, $f(\varphi)$ may be presented by the Fourier series

$$f(\varphi) = \frac{1}{2} a_0 + \sum_{l=1}^{\infty} a_l \cos 6l\varphi$$

For stress-state type dependent materials acceptable refinements can be achieved with the first three terms of the series. Thus, in the first approximation it is assumed that

$$f(\varphi) = A(1 + a \cos 3\varphi + b \cos^2 3\varphi)$$  \hspace{1cm} (10)

Material constants $A$, $a$ and $b$ can be evaluated on the basis of two typical uniaxial tests and a shear test. When only uniaxial test data in tension and compression are available, then $f(\varphi)$ may be taken in the form

$$f(\varphi) = A(1 + a \cos 3\varphi)$$  \hspace{1cm} (11)

4.2 Failure and initial cracking criteria

Referring to the approximations of the material function $f(\varphi)$, failure criterion (2) may be expressed in the form:-

$$\Sigma = \sqrt{\frac{3}{2}} (1 + a \cos 3\varphi + b \cos^2 3\varphi) \tau = k$$  \hspace{1cm} (12)

or

$$\Sigma = \sqrt{\frac{3}{2}} (1 + a \cos 3\varphi) \tau = k$$  \hspace{1cm} (13)

where the product $BA$ is arbitrarily taken as $\sqrt{\frac{3}{2}}$. 

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On the basis of known uniaxial strengths $f_c$ and $f_t$ and pure shear strength $f_0$, the parameters $a$, $b$ and $k$ can now be evaluated by the following equation:

\[
\begin{bmatrix}
-f_t & -f_t & \sqrt{3} \\
-f_c & -f_c & \sqrt{3} \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
k
\end{bmatrix}
= \begin{bmatrix}
f_t \\
f_c \\
f_0
\end{bmatrix}
\]  
(14)

or, in the case of criterion (11):

\[
\begin{bmatrix}
-f_t & \sqrt{3} \\
f_c & \sqrt{3} \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
a \\
k
\end{bmatrix}
= \begin{bmatrix}
f_t \\
f_c
\end{bmatrix}
\]  
(15)

To perform a comparison between the results obtained using the proposed criteria and experimental data, reference is made to the mechanical characteristics of concrete block masonry determined experimentally by Khatab and Drysdale [10]. This allows a quantitative comparison of the experimental and analytical results of the strength of masonry under the biaxial tension-compression state of stress. To determine the criteria parameters, the following values were assumed:

$f_t = 0.65$ MPa, $f_c = -9.5$ MPa, $f_0 = 1.2$ MPa.

Material constants are found to be: $a = 1.7$, $b = 0.47$, $k = 1.2$ MPa.

Assuming that the initial cracking criterion has a similar functional form to (3), then parameters of the material function $f(\varphi)$ can be determined using (14) and (15) on the basis of known uniaxial proportional limiting stresses $f'_t$ and $f'_c$ and pure shear proportional limit $f'_0$. Idealised intersection of the initial microcracking and failure surfaces with the plane $\sigma_3 = 0$ are given in Figure 2.
4.3 Stress-strain relationships

In order to complete the proposed model of masonry behaviour the material functions included in the stress-strain relationships (8) are used. It was assumed that the specific distortion energy $W(\Sigma)$ may be taken as a polynomial form of second degree of the effective stress $\Sigma$. The following form is used:

$$W(\Sigma) = \frac{3}{4G}(1 + a \cos 3\varphi + b \cos^2 3\varphi) \tau^2$$

where $G$, $a$ and $b$ are material constants. It follows from the condition of homogeneity that the dilatation energy function $\Phi_i(\sigma)$, also should be taken a second degree of the invariant $\sigma$, i.e.

$$\Phi_i(\sigma) = \frac{1}{2K}(1 + c Sgn(\sigma) \sigma^2$$

where $K$ and $c$ are material moduli. Then, applying the relationships stated in (8), the stress-strain relationships of masonry can be expressed as:-

$$\varepsilon_{kl} = \frac{1}{3K} (1 + a Sgn(\sigma) \sigma \delta_{kl} + \frac{1}{2G} (1 + b \cos 3\varphi + c \cos^2 3\varphi) \sigma^2$$

where $K$, $G$, $a$, $b$, $c$ are independent parameters determined from standard tests in uniaxial tension, compression and shear:

$$\frac{1}{2G} = \frac{1}{2G_0}, \quad b = G_0 \left(\frac{1 + \nu_t}{E_t} - \frac{1 + \nu_c}{E_c}\right), \quad c = G_0 \left(\frac{1 + \nu_t}{E_t} + \frac{1 + \nu_c}{E_c}\right) - 1,$$

$$\frac{1}{3K} = \frac{1}{2} \left(\frac{1 - 2\nu_t}{E_t} + \frac{1 - 2\nu_c}{E_c}\right), \quad a = \frac{E_c (1 - 2\nu_t) - E_t (1 - 2\nu_c)}{E_t (1 - 2\nu_t) + E_c (1 - 2\nu_c)}$$

where $G_0$ denotes the modulus of pure shear. Note, that if $G_0$ is not an independent parameter, i.e., the properties of the material in shear are defined by properties in tension and compression, then. it will follow from (16) that:

$$c = 0, \quad b = \frac{E_c (1 - \nu_t) - E_t (1 - \nu_c)}{E_t (1 - \nu_t) + E_c (1 - \nu_c)}, \quad \frac{1}{2G} = \frac{1}{2} \left(\frac{1 + \nu_t}{E_t} + \frac{1 + \nu_c}{E_c}\right)$$

If the material has equal uniaxial properties with the elastic moduli $E$ and $\nu$, but pure shear is an independent stress state, then:

$$\frac{1}{2G} = \frac{1}{2G_0}, \quad b = 0, \quad c = \frac{2G_0 (1 + \nu)}{E} - 1, \quad \frac{1}{3K} = \frac{1 - 2\nu}{E}, \quad a = 0$$

For the conventional elastic material, from (16) it follows that:
\[ \frac{1}{2G} = \frac{1 + \nu}{E}, \quad \frac{1}{3K} = \frac{1 - 2\nu}{E}, \quad a = b = c = 0 \]

For further elaboration of the model, the linear elastic behaviour in the first stage of loading is assumed,

\[ \epsilon_{kl} = \frac{1}{3K} \sigma_{kl} + \frac{1}{2G} \sigma^0_{kl}, \quad k, l = 1, 2, 3 \]

where constants \( K, G \) are taken as in (16), thus satisfying the compatibility condition at the point of initial cracking.

5. CONCLUSIONS

A novel constitutive model representing the behaviour of masonry is presented. A principal feature of the model is the stress-state type dependence of strength and the elastic properties. Further development of the model by the authors will cater for the anisotropic behaviour of masonry.

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7. REFERENCES


