12TH INTERNATIONAL
BRICK/BLOCK
Masonry
CONFERENCE

MADRID-SPAIN
25-28 June 2000

Shr

SIMPLE DESIGN PROCEDURES FOR MASONRY ARCHES

N. G. Shrive[1], M. M. Reda[2], and A. Huizer[3]

[1]Killam Memorial Professor, Department of Civil Engineering, University of Calgary, Calgary, Alberta, Canada.

[2]Structural Engineer, Ph.D., Campbell Woodall and Associates, Calgary, Alberta, Canada.

[3]Senior Instructor Emeritus, Dept of Civil Engineering, University of Calgary, Alberta, Canada.

ABSTRACT

The masonry arch has declined rapidly as a form of construction in the last few decades. This is in part due to perceived cost of initial construction, but also in part because of the lack of education in arch design. Even recent textbooks on structural masonry omit arch design as a subject. The method by which arch design is taught at the University of Calgary is described. The funicular polygon and Bow's notation must be explained in order to introduce the concept of line of thrust and the graphical method of Curtin et al. (1). Analysis of 3-pin and 2-pin arches is covered quickly with emphasis being placed on the analysis of fixed arches and the inherent assumptions. Given the comfort of modern students with computers two methods of programming are provided to students. The first utilizes a spreadsheet to solve for the unknowns, by dividing the arch into discrete segments and summing contributions to integral equations. The second utilizes MathCad® and has the advantage that graphs can be plotted of the thrust in relation to the kern of the arch. This returns to the visual aspect of the graphical method and reveals where careful analysis of arch stresses is required.

Key words: *Arch design, structural masonry, spreadsheets, exact analysis graphical design and thrust line*

1. INTRODUCTION

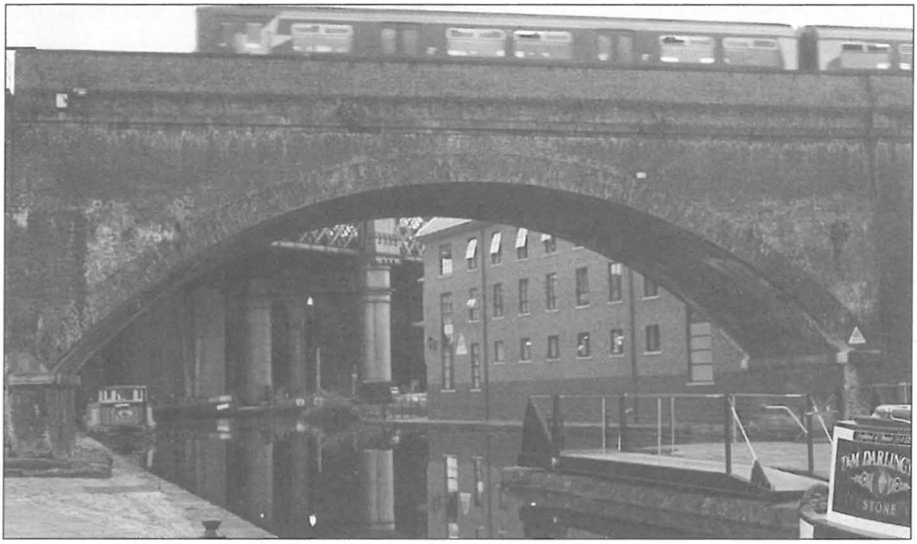
Masonry arch construction has been used for about 5000 years beginning with the Chinese and passing through the Babylonians, Egyptians, Greeks, Romans and Normans to modern times. The ubiquitous presence of the masonry arch in vaulted roofs, aqueducts, and bridges attests to its inherent strength, tolerance of movement, and ease of construction. Many examples, ancient, medieval and relatively modern, exist in many areas of the world as demonstrated in Figures 1 and 2.

The versatility of the arch can be seen in the many shapes and sizes seen over time. For example Roman semicircular, Norman pointed, Tudor and segmental arches all have distinct, differing shapes. Like a suspension bridge that is in concert with its surroundings, an arch is aesthetically pleasing. Hence engineers should be able to design an arch to satisfy architectural needs. It is very unfortunate therefore that modern structural education typically emphasizes rectangular frames and space structures with beams and columns. Reflecting the modern emphasis on straight lines is the omission of arch design from recent masonry texts (2, 3 and 4). This has led to the addition of more materials to masonry construction, sometimes with detrimental results. For example, reinforced concrete lintels in a masonry wall can lead to unsightly cracks caused by the differential movement between the two materials.

Figure 1. Masonry Arches in the Ming tombs outside Beijing, China.

Early arches were no doubt designed by rule-of-thumb, learned from practical experience and passed from one master builder to another. Engineers studied these rules and existing, successful arches to the extent that by Victorian times,

Figure 2. Victorian masonry arch bridge in Manchester, UK.



arches were designed by calculation(6). Masonry arches are a particularly robust structural form and can tolerate substantial movement in abutment and spandrel walls without failure (5). Considerable increases in load can also often be withstood. A typical example is Brunel's flat arches over the Thames at Maidenhead (6), now subject to much heavier trains moving at much faster speeds than originally envisaged.

One reason for the dearth of modern masonry arch construction may be perceived cost. Certainly a masonry arch is likely to cost more than a precast concrete girder to construct. However life-cycle costing should be considered, including maintenance cost and durability. The many arch bridges which are still performing more than adequately compared to the immense deficit now being incurred in the repair of reinforced concrete structures, should be sending a strong message to municipal engineers.

Arch design is considered to be complex because of the curved shape. However, this need not to be the case. Within the structural masonry course taught at The University of Calgary, arch design is covered in a straight forward manner, avoiding complex analysis and making use of software available on PCs. We hope that arch design can be introduced to a wider audience using this approach.

2. TEACHING SEQUENCE

Arches are introduced with a description of architectural terms; keystone, intrados, extrados springing point, span, rise, etc. This is followed by a description of the structural principles, and an introduction to the concept of the "thrust line".

This is a new concept for students who have not been exposed to graphical techniques but have been given a diet of numerical analysis of trusses, frames, beams and slabs, through the force and displacement methods.

The assumptions of mathematical analysis of arches are then examined:

(a) The abutments do not move.
What are the effects of the abutments spreading (3 pin arch), pushing together (3 pin arch) or settling differentially?

(b) The arch profile does not change.
What are the effects of workmanship; thermal, moisture and creep movements?

(c) Masonry is isotropic.
What are the effects of masonry anisotropy that need to be remembered?

(d) Arch loading is uniform.
What are effects of other loading cases?

The graphical design procedure described in Curtin et al. (1) is then introduced. The geometry of the arch (span, rise and shape) must be chosen. Then a first estimate on the arch thickness is obtained from the various formulae developed by Victorian-era engineers. The concepts and methods of the funicular polygon and Bow's notation are explained as fundamental to understanding the graphical approach. Students have not been exposed to these methods in earlier courses. Also, with the modern reliance and comfort with computers, the idea of drawing the line of thrust on an arch profile by hand is alien as a method of design. Students are not at all comfortable with this idea. However, it is important that students understand the method so that they realize how the line of thrust superimposed on the profile reveals where the danger spots are and thus where more careful analysis is required. It is also important to realize that this visual method will allow them to hone in on a suitable arch profile quite quickly. That they can do this with a ruler and a pencil in the absence of a PC is a by-product of understanding how an arch functions, and that differing profiles will perform differently under a given loading scheme. Theoretical analyses of arches are then presented: 3-pin, 2-pin and lastly fixed arches.

3. ARCH ANALYSIS

3-pin arch and 2-pin arch analyses are dealt with rapidly but include parabolic arch equations and how to obtain local axial and shear components. With the 2-pin arch, the effect of ignoring axial deformation is considered, particularly with flat arches. Students are shown how to assess axial deformation in comparison to bending deformation. It is in the analysis of fixed arches that we have made progress in the use of software. A fixed arch can be analyzed using the configuration

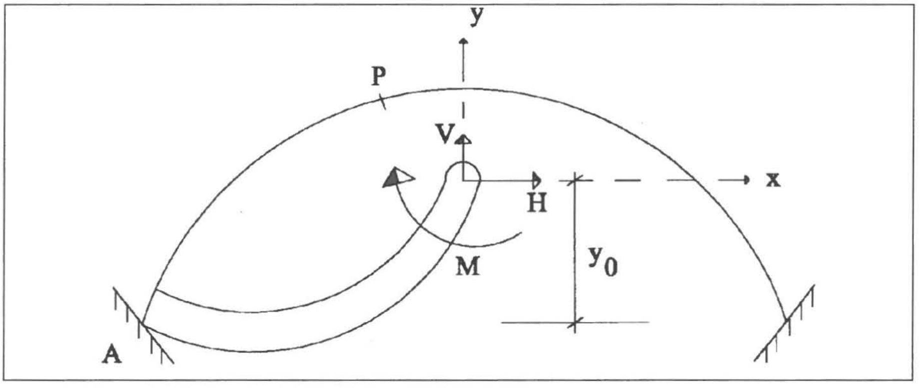
shown in Figure (3). An infinitely rigid arm is assumed to connect the left hand abutment "A" to the elastic centre, "the centroid" of the arch. The three load actions at the centroid are then defined by equations (1) to (3)

$$M = \frac{\int M_p \frac{ds}{EI}}{\int \frac{ds}{EI}} \tag{1}$$

$$V = \frac{-\int M_p x \frac{ds}{EI}}{\int x^2 \frac{ds}{EI}} \tag{2}$$

$$H = \frac{-\int M_p y \frac{ds}{EI}}{\int y^2 \frac{ds}{EI}} \tag{3}$$

Figure 3. A fixed symmetrical arch with an infinitely rigid arm joining the abutment A to the elastic centroid (Origin of coordinates x and y).



The formulae are accurate so long as axial and shear deformations can be ignored. The integrals are taken over the complete arc length of the arch, Δs being an increment of arc length. x and y are coordinates within the system shown in Figure (3). M_p is the moment effect at a general point P on the arch profile due to the loading (dead and live) between A and P. The elastic centroid is determined using the integration in equation (4).

$$y_0 = \frac{\int y' \frac{ds}{EI}}{\int \frac{ds}{EI}} \tag{4}$$

where, in this case, y' is a coordinate system defined from any point on the line through the springing points. A different horizontal line could be used (e.g. through the crown) with y_o then defining the distance from that line to the level of the elastic centre. The equations for y_0, M, V, and H can be programmed into software like MathCad® or can be solved by dividing the arch into discrete segments and performing summations in a spreadsheet.

4. SPREADSHEET METHOD

For simplicity it is easiest to divide the arch into segments of equal horizontal length. The (x',y') coordinates of the centre and edge points of each segment are then defined: again for simplicity, it is convenient to use the left springing point as the origin of the (x',y') axes. The first part of the spreadsheet can then be established with)x' and)y' (the increment in x' and y' coordinates for each segment) required as input. Column headings are shown in Table (1).)x' and)y' can be calculated from the arch equation if the shape can be expressed mathematically, with input therefore being just the span,rise and shape of the arch.

Table 1. Part one of spread sheet calculations for the design example presented

Section	$\Delta x'$	$\Delta y'$	Δs	h	Δw	$\Delta s/EI$	$y'_{(c.g)}\ y'$	(ds/EI)
1	1	0.94	1.37	0.4443	14.008	1	0.484	0.4844
2	1	0.81	1.29	0.4353	12.899	1	1.359	1.3594
3	1	0.69	1.21	0.4267	11.908	1	2.109	2.1094
4	1	0.56	1.15	0.4188	11.050	1	2.734	2.7344
5	1	0.44	1.09	0.4118	10.339	1	3.234	3.2344
6	1	0.31	1.05	0.4063	9.790	1	3.609	3.6094
7	1	0.19	1.02	0.4023	9.414	1	3.859	3.8594
8	1	0.06	1.00	0.4003	9.224	1	3.984	3.9844
9	1	-0.06	1.00	0.4003	9.224	1	3.984	3.9844
10	1	-0.19	1.02	0.4023	9.414	1	3.859	3.8594
11	1	-0.31	1.05	0.4063	9.790	1	3.609	3.6094
12	1	-0.44	1.09	0.4118	10.339	1	3.234	3.2344
13	1	-0.56	1.15	0.4188	11.050	1	2.734	2.7344
14	1	-0.69	1.21	0.4267	11.908	1	2.109	2.1094
15	1	-0.81	1.29	0.4353	12.899	1	1.359	1.3594
16	1	-0.94	1.37	0.4443	14.008	1	0.484	0.4844
SUM						**16.0**		**42.750**

The increment in arc length, $\Delta S = \sqrt{(\Delta x')^2 + (\Delta y')^2}$ h is the arch thickness which may vary around the arch. Varying h will imply varying I, required for the calculations of)S/EI. y is the location of the centre of the segment in the vertical direction. The dead weight should include the weight of the masonry plus the weight of any fill above that segment of the arch. Summations are performed on the)S/EI and y')S/EI columns. The latter sum divided by the former provides the required value of y_0 as in equation (4). The second part of the spreadsheet can now be established as shown in Table (2).

Table 2. Part two of spread sheet calculationsfor the design example presented

Section	x	y	$y^2(\Delta s/EI)$	$x^2(\Delta s/EI)$	M_p	$M_p(\Delta s/EI)$	$M_py(\Delta s/EI)$	$M_px(\Delta s/EI)$
1	-7.5	-2.19	4.785	56.25	5.00	5.002	-10.942	-37.516
2	-6.5	-1.31	1.723	42.25	29.12	29.121	-38.221	-189.285
3	-5.5	-0.56	0.316	30.25	79.90	79.904	-44.946	-439.474
4	-4.5	0.06	0.004	20.25	154.61	154.613	9.663	-695.759
5	-3.5	0.56	0.316	12.25	252.39	252.390	141.970	-883.367
6	-2.5	0.94	0.879	6.25	372.53	372.527	349.244	-931.319
7	-1.5	1.19	1.410	2.25	514.48	514.476	610.940	-771.714
8	-0.5	1.31	1.723	0.25	677.86	677.862	889.693	-338.931
9	0.5	1.31	1.723	0.25	862.50	862.495	1132.025	431.248
10	1.5	1.19	1.410	2.25	1068.38	1068.377	1268.697	1602.565
11	2.5	0.94	0.879	6.25	1295.70	1295.696	1214.715	3239.239
12	3.5	0.56	0.316	12.25	1544.83	1544.826	868.965	5406.890
13	4.5	0.06	0.004	20.25	1816.32	1816.316	113.520	8173.420
14	5.5	-0.56	0.316	30.25	2110.87	2110.874	-1187.367	11609.809
15	6.5	-1.31	1.723	42.25	2429.36	2429.358	-3188.532	15790.827
16	7.5	-2.19	4.785	56.25	2772.76	2772.755	-6065.402	20795.666
SUM			**22.313**	**340.000**		**15986.6**	**-3935.979**	**62762.299**

x and y are now the coordinates of the centres of the segments defined in the co-ordinate system with the elastic centre as origin. M_p the moment effect at the nth segment centroid due to the loading between A and that centroid, is given by Equation (5).

$$M_p = \frac{q\left[\left(n-1\right)\Delta x + \frac{\Delta x}{2}\right]^2}{2} + \sum_{i=1}^{n} \left(D.L.\right)_i * \left(x'_{centroidn} - x'_{centroidi}\right) \tag{5}$$

where q is the uniformly distributed live load. Clearly the first part of equation (5) depends on the load case being analyzed; in this case a full span uniformly distributed load. The following columns are summed: y^2 *()s/EI)*, x^2*()s/EI)*, M_P*()s/EI)*, M_py*()s/EI)*, and M_px*()s/EI)*. Values of *M, V* and *H* at the elastic centroid are then determined according to equations (1,2 and 3) presented earlier. For a symmetrically loaded arch, *V* should be half the total vertical force. *H* can be checked against the value obtained by assuming the arch is 3-pin. The moment at any point of the arch can be found from Equation (6).

$$M\left(x\right) = M + Vx - Hy - M_P \tag{6}$$

If the thrust is found at the same point, the eccentricity can be determined and compared with kern values to check for possible cracking.

5. MATHCAD PROGRAM

A computer program using MathCad® was developed to provide exact analysis of the arch. Here data entry involves the span and rise of the arch, together with the

arch equation. As above, the self weight of the arch and fill will be required, together with the live load. The integral equations to determine y_0, M, V and H can be programmed directly. The integration will basically depend on the function governing the change of the arch thickness along the arch length. This function will be involved when relating)s to)x and when determining moments. The current program allows two cases of live loading: uniformly distributed load on either the full arch span or on half the span. The program also provides the user with the capability of incorporating axial deformation when a flat arch is being analyzed. Once calculations are complete, the bending moment diagram can be plotted. Also the line of thrust can be plotted with respect to the kern of the arch.

6. EXAMPLE

Both methods have been used to determine the reactions for a parabolic arch of span 16 m and rise 4 m. For one metre width of the arch the live load is 12 kN/m: the weight density of the masonry is 23 kN/m^3. h_0 at the crown is 0.4 m. The section of the arch is to change with ()s/EI)=()x/EI_0) where EI_0 is constant. The section change is therefore given by Equation (7).

$$h = h_0 \sqrt[3]{\frac{\Delta s}{\Delta x}} \tag{7}$$

The spread sheet calculations of this example are given in Tables (1) and (2). In Table (3) the results of the two methods are compared. The arch was divided into 16 segments in the spreadsheet method.

Table 3. Results of arch analysis comparing the spreadsheet and the MathCad® methods

Straining action	Spreadsheet method	MathCad® method
M (kN.m)	999.2	1001.1
V (kN)	184.6	184.7
H (kN)	176.4	176.3

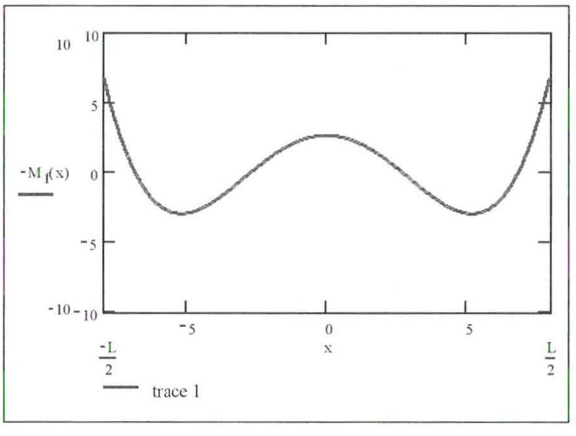


Figure 4. Bending moment diagram due to dead load and full span live load.

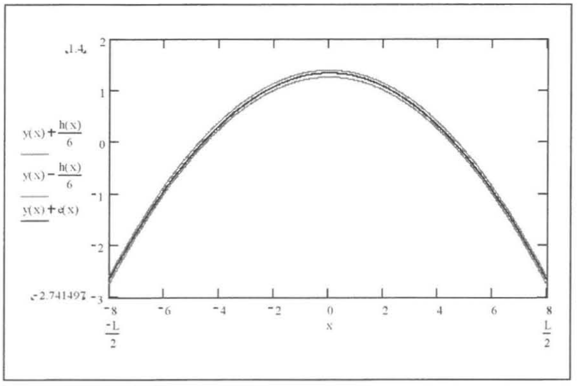


Figure 5. Line of thrust with respect to the kern of the arch (Full span live load).

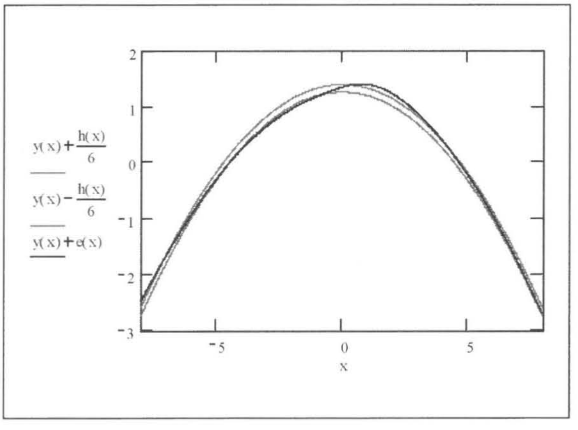


Figure 6. Line of thrust with respect to the kern of the arch (Half span live load). The zone to the right of the crown needs further analysis.

The advantage of the MathCad® analysis can be seen in Figures (4), (5), and (6) where the bending moment (Figure 4) and line of thrust with respect to the kern of the arch (Figure 5) are plotted for the case of uniformly distributed live load over the entire span of the arch. In Figure (6), the line of thrust is shown for the case where only the left hand half of the arch is subjected to the uniformly distributed live load.

7. CONCLUSION

Arch design should not be ignored in masonry design courses. Arches can be designed using software available on and for PCs, providing both numerical and graphical information which can be interpreted easily by students. Careful analysis of selected points on the arch can then be performed. Arch shapes can be assessed through repeated analysis until a suitable one is obtained for the loading expected. The use of computers is seen as very beneficial for modern students who are comfortable with computer analysis of structures.

8. REFERENCES

1. Curtin, W. G., Shaw, G., Beck, J. K., amd Bray, W. A., "Structural Masonry Designers' Manual" Granada Publishing, London, 1982, 498 pp.
2. Drysdale, R. G, Hamid, A. A., and Baker, L., "Masonry Structures: Behaviour and Design", 2nd Edition, The Masonry Society, Boulder Colorado, 1999, 888 pp.
3. Glanville, J. I, Hatzinikolas, M.A., Ben-Omran, H. A., "Engineered Masonry Design; Limit States Design", Winston House, Winnipeg, Manitoba, 1996, 368 pp.
4. Hendry, A. W., Sinha, B. P., and Davis, S. R., "An Introduction to Load Bearing Masonry Brickwork Design", Ellis Horwood Ltd, John Wiley and Sons, 1981, 184 pp.
5. Jones, C.J.F.P., "Defects Originating in the Ground", In "The Maintenance of Brick and Stone Masonry Structures", Ed., Sowden, A. M., E & F.N.Spon, pp. 141-156.
6. Owen, J. B. B., "Arch Bridges" In "The Works of Isambard Kingdom Brunel: An Engineering Appreciation", Ed: A. Pugsley, Cambridge University Press, 1980, pp. 89-106.