13th International Brick and
Block Masonry Conference
Amsterdam, July 4-7, 2004

LIMIT ANALYSIS FOR THREE-DIMENSIONAL STONE MASONRY STRUCTURES WITH FRICTION

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Abstract

The paper deals with the limit analysis of structures made of rigid blocks interacting through no-tension and frictional contact surfaces performed using mathematical programming methods. The presence of friction makes the problem non-linear and non-convex. It is shown that a proper choice of the initial guess for the variables of the non-linear program enables the convergence to the optimal solution. The initial variables are evaluated as the outputs of a linear program performing the limit analysis of the same assembly of blocks with dilatancy along the joints, instead of friction. Various analyses show the validity of the procedure and the effectiveness of the limit analysis approach for studying the collapse behaviour of real three-dimensional stone masonry structures.

Key Words

Block-Masonry Structures, Limit Analysis, Mathematical Programming, Friction

1 Introduction

In this work a computer procedure for the detection of the collapse behaviour of stone masonry assemblages, based on non-standard limit analysis method, is proposed. The procedure elaborated provides a computational tool suitable to be used in practical structural analyses for assessing the safety of ancient masonry buildings, with particular references to the seismic actions.

In spite of the many sophisticated tools of analysis available for the analysis of masonry structures at this time, most of them based on Finite Element codes, appropriate and simple tools for the structural analysis of ancient masonries, generally made of stones dry assembled together or with joints filled by poor and scattered mortar, are still lacking. In particular, non-linear finite elements approaches, both for generalised homogeneous classical and micropolar continua (Lourenço *et al.* 1998; Trovalusci, Masiani 2003) and for discontinuous interface models (Baggio, Trovalusci

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1995), that properly account for the most important features of such discontinuous and heterogeneous structures, appear difficult to be used because of the advanced knowledge of the solution techniques required to the practitioners, the uncertainty about the definition for the material mechanical properties, the large time required for the model characterisation and the analyses. The more practical linear elastic finite elements approaches instead, completely fail when evaluating the stress fields.

Besides the mechanical properties of the constituents, the incapability to carry tension along the joints and the friction at the interfaces, the strength of this kind of masonry strongly depends on the shape, the size and the disposition of the stones. Moreover, at the collapse, the rigid displacements of the elements are predominant with respect to their deformability. The extremely variable nature of ancient masonry, often made of blocks of various sizes assembled together according to various textures without, or with poor and scattered, mortar makes the structural problem very difficult to be fulfilled.

The discrete modelling still appears the most effective approach to study masonry structures taking into account the important features related to their discontinuous and heterogeneous nature, in a word on their particular ‘building rule’ (Baggio, Trovalusci, 1995). In this framework, in order to assess the structural safety, non-standard limit analysis approach, based on the ideas contained in the work of Drucker (1954), proves to be one of the most effective approaches.

Although the development of computer optimisation routines offers now the chance of performing non-standard limit analysis, the applicability of this approach to real masonry structures, with many degrees of freedom, has been for long time compromised due to the large size of the corresponding numerical problems. The presence of friction moreover, involves non-associated flow rules that make the mathematical programming problem non linear and non convex and in general difficult to be solved.

In some recent works however, effective computational strategies to cope with this problem has been proposed (Baggio, Trovalusci 1998, 2000, Ferris, Tin-Loy, 2001). In particular, in this work we follow the strategy proposed in (Baggio, Trovalusci, 2000) based on previous solution of a linearised programming problem used to select a proper initial estimate for the variables of the non-linear program that enables the convergence to the optimal solution. The initial variables for the NLP are evaluated as the output of the linear program corresponding to the limit analysis of the same assembly of blocks with dilatancy along the joints, instead of friction.

In this way it has been possible to analyse two and three-dimensional structures of practical interest made of many stones of various size and disposition, evaluating the collapse loads and the collapse mechanisms in presence of horizontal body forces that simulate statically the seismic action.

This approach combines the advantages of proper mechanical models for stone masonry assemblies with frictional joints, sufficient statical information about the safety of constructions and relative easiness in casting into a computational tool. Moreover, only the actual geometry of the assembly and the friction at the interfaces must be defined for the calculations, generally without uncertainties.

Notation

N	number of blocks of the systems
M	number of interfaces (joints)
$\mathbf{u}$	vector of blocks displacements (generalised displacements, dim$\mathbf{u}$ $6N$)
ε	vector of relative displacements and rotations between joints (generalised strains, dimε $6M$
$\mathbf{B}$	kinematic matrix (dim$\mathbf{B}$ $6M$x$6N$)
λ	vector of modes of failure at interfaces ("plastic multipliers", dimλ $18M$)
$\mathbf{M}$	matrix of coefficient of modes of failure (dim$\mathbf{M}$ $6M$x$18M$)
$\mathbf{f}_0$	vector of "dead" loads (generalised permanent actions $6N$),
α	load multiplier ($\alpha \geq 0$)
$\alpha\mathbf{f}_L$	vector of "live" loads (dim$\mathbf{f}_L$ $6N$)
σ	vector of interactions (generalised stress, dimσ $6M$)
$\mathbf{N}^T$	"gradient matrix" (dim$\mathbf{N}$ $18M$x$6M$)

2 Limit analysis for systems of blocks with no-tension and frictional interfaces

2.1 Governing equations

A computer procedure to study the collapse behaviour of brick/block masonry structures, with particular reference to ancient masonry structures made of stones dry assembled together or with joints filled by poor and scattered mortar, is proposed. The mechanical model adopted is a discrete model in which the mechanical properties of the constituents, the incapability to carry tension along the joints and the friction at the interfaces are accounted for as well as the geometry of the assembly (shape, size and disposition of stones). As, at the collapse, the rigid displacements of the elements are predominant, with respect to their deformability, the blocks are considered rigid interacting through no-tension and frictional interfaces.
The proposed procedure regards systems made of N rigid parallelepiped blocks interacting through M no-tension and frictional contact surfaces. The blocks can translate and rotate (vector $\mathbf{u}$) and as strain measures of the assembly the relative displacements and the relative rotations between blocks are assumed, ordered in the vector ε. This vector is expressed as linear combination of 18 elementary modes of failure at each interface represented by the vector λ (the contact surface is described by a polygon of 8 edges, the modes of failure considered are: 8 rotations around the edges, 8 slidings in four different directions; 2 in-plane rotations).
The blocks are subjected to permanent loads, $\mathbf{f}_0$, and to the action of proportionally increasing loads, $\mathbf{f}_L$, governed by a single non-negative parameter, α. They can interact through forces and couples, ordered in the vector σ. As the joints cannot carry tension and the tangential forces, as well as the in-plane couples, at the interfaces are limited by the frictional strength, bounds on the stress components must be posed. These bounds define a piece-wise linear yield domain.
The governing equations of the problem are:
the kinematics compatibility equations

$$\mathbf{Bu} = \boldsymbol{\varepsilon}\text{, with} \tag{1}$$
$$\varepsilon = \mathbf{M}\lambda\,; \tag{2}$$

the balance equations

$$\mathbf{B}^{\mathrm{T}}\sigma = \mathbf{f}_0 + \alpha\mathbf{f}_L\,; \tag{3}$$

the yield inequalities (“flow-rule”)

$$\boldsymbol{\varphi} = \mathbf{N}^{\mathrm{T}}\boldsymbol{\sigma} \le \mathbf{0}\,; \tag{4}$$

and the “complementarity” condition

$$\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\lambda} = \mathbf{0}\,. \tag{5}$$

The last condition represents the fact that to the “activation” of a single face of the yield domain, in case of limit stress, a relative displacement corresponds. This displacement is normal to the face in case of limit bending while, in case of limit shear or limit torque is not normal.

2.2 Mathematical programming for the detection of the collapse load

As widely acknowledged, the problem of evaluating the collapse load and the collapse mechanism of lagrangian systems of elements interacting through no-tension and frictional contact surfaces has not unique solution, both in terms of load multiplier and in terms of contact actions in the joints. However, for the collapse load multiplier lower and upper bounds can be found (Drucker 1954). In order to asses the structural safety, we then elaborated a code that, according to the theorems of limit analysis for non-standard systems, searches the minimum (safe) collapse load parameter, α^{lim}, corresponding to both kinematically and statically admissible states, (1)-(5), using the methods of mathematical programming (Baggio, Trovalusci 2000).

Standard algebraic manipulations of the governing equations (1)-(5) lead to the following mathematical programming

Determine min F(α)=α, subjected to:

$$(\mathbf{A}_0\mathbf{N})^{\mathrm{T}}\mathbf{f}_0 + \alpha(\mathbf{A}_0\mathbf{N})^{\mathrm{T}}\mathbf{f}_{\mathrm{L}} - (\mathbf{AN})^{\mathrm{T}}\sigma_2 \ge \mathbf{0} \qquad \text{(balance, yield domain)} \quad (7)$$

$$\mathbf{AM}\boldsymbol{\lambda} = 0 \qquad \text{(kinematic compatibility, “flow-rule”)} \quad (8)$$

$$\boldsymbol{\lambda}^{\mathrm{T}}(\mathbf{A}_0\mathbf{M})^{\mathrm{T}}\mathbf{f}_{\mathrm{L}} - 1 = \mathbf{0} \qquad \text{(positive live load)} \quad (9)$$

$$\boldsymbol{\lambda}^{\mathrm{T}}(\mathbf{A}_0\mathbf{N})^{\mathrm{T}}\mathbf{f}_0 + \alpha\boldsymbol{\lambda}^{\mathrm{T}}(\mathbf{A}_0\mathbf{N})^{\mathrm{T}}\mathbf{f}_{\mathrm{L}} - \boldsymbol{\lambda}^{\mathrm{T}}(\mathbf{AN})^{\mathrm{T}}\sigma_2 = \mathbf{0} \qquad \text{complementarity} \quad (10)$$

with the unknowns $\alpha \ge 0$, $\lambda \ge \mathbf{0}$ e σ_2, where σ_2 is the vector of the statically undetermined internal actions, $\mathbf{A}$ and $\mathbf{A}_0$, are matrices depending on the geometry of the assembly, with $\mathbf{A}_0$ the inverse of the maximum rank kinematical matrix.
The non-linear equality constraint (10), due to the presence of non associative flow rules ($\mathbf{M} \neq \mathbf{N}$), makes the problem non-linear and non-convex (NLP).
The solution of this problem is a hard task from a mathematical and numerical point of view. In non-linear and non-convex programming problems in fact, it is not guaranteed that an optimal solution is a global minimum, even if the Khun-Tucker optimality conditions are verified. Moreover, the convergence to the solution strongly depends on the choice of the initial estimate for the unknowns of the program.

2.3. A strategy for the solution of non-linear programming

The large numerical difficulties related to the solution of the NLP and the strong dependence on the computational possibilities of the computers limited for many years the practical potentiality of the limit analysis approach, especially for high degrees of freedom structures. Anyway, the computer procedure elaborated facilitates, and in some cases makes possible, the convergence to the solution of the NLP.

This procedure is based on the obvious observation that if a good initial estimate for the unknowns of the NLP can be found, the program easily converges to an optimal solution. The initial variables for the NLP problem are here evaluated as the outputs of a programming problem corresponding to the limit analysis of the block system in which frictional joints are replaced with dilatant joints. In this case, the problem proves to be linear and easy to be solved. In fact, if sliding between two blocks is allowed only when blocks move in the direction normal to the contact surface (dilatancy), along a direction defined by the friction angle, the strain vector, ε, is always normal to the given yield surface and the flow rule is associated ($\mathbf{M}=\mathbf{N}$).

In such circumstance a standard limit analysis approach can be used to detect the collapse load, α^+, and the collapse mechanism of the assembly with dilatant interfaces, with the given yield domain and associated flow rule. Using the upper bound approach for instance, the linear programming (LP) problem to be solved writes

Determine min $\{\alpha=\lambda^T(\mathbf{A}_0\mathbf{N})^T\mathbf{f}_0\}$ subjected to:

$$\mathbf{AN}\lambda = \mathbf{0} \qquad \text{(kinematic compatibility, flow rule)} \qquad (11)$$

$$\lambda^T(\mathbf{A}_0\mathbf{N})^T\mathbf{f}_L - 1 = \mathbf{0} \qquad \text{(positive live work)} \qquad (12)$$

with the unknowns $\lambda \geq \mathbf{0}$. The remaining statical unknowns σ_2, are obtained as dual unknowns.

The collapse multiplier, α^+, of a system with a given yield domain and associated flow rule is an upper bound for the class of the collapse multipliers of the same system with the same yield domain but non-associated flow rule. Moreover, the LP solution, α^+, λ^+ and $\sigma_2{}^+$, gives a "quasi-feasible" point for the NLP problem. In fact all the constraints (7)-(10), except the constraint (8), are satisfied. This solution can be used as good initial estimate for the unknowns α, λ and σ_2. This enables the convergence to the optimal solution of the NLP problem. By slightly perturbing this initial guess, it is possible to check if the optimal point is a global minimum.

The proposed algorithm works in two steps.

Step 1:

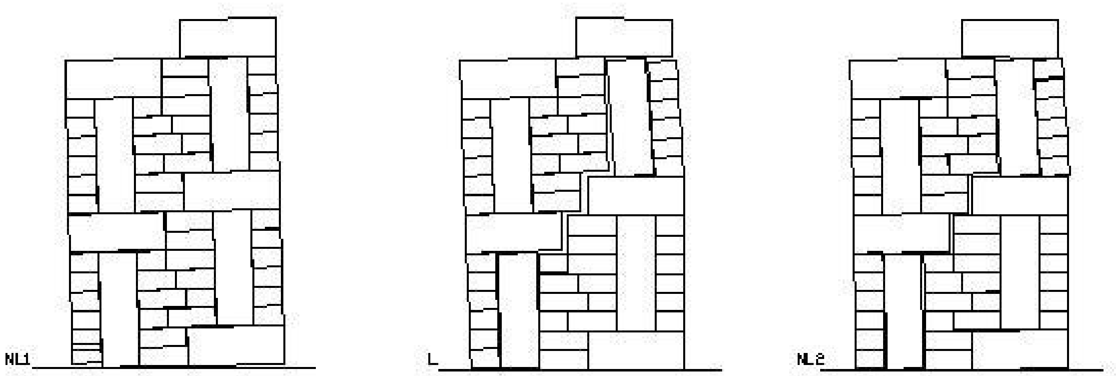
- generates the geometrical and the material (friction angle) input data of the assembly;
- solves the linear programming problem (LP), for the assembly with dilatant interfaces, giving the primal, α^+, λ^+, and the dual, σ^+, unknowns.
- selects $\sigma_2{}^+$ (contact actions in the statically undetermined joints);

Step 2:

- assumes α^+, λ^+ and $\sigma_2{}^+$, as initial estimate for the variables of the NLP and gives the optimal solution $\alpha^{\lim}$, λ, and σ_2 for the actual assembly with frictional interfaces;
- slightly perturbs the initial guess, α^+ λ^+, and σ^+, to check local optimal points.

3 Collapse mechanisms of two and three-dimensional assemblages

In the previous session it has been asserted that a proper choice of the initial guess for the variables of the non-linear program enables the convergence to the optimal solution. If the initial variables are evaluated as the outputs of a linear program corresponding to the limit analysis of the same assembly of blocks with dilatancy along the joints, instead of friction, it can be shown that convergence of the NLP to the optimal solution can be often ensured.
In this way it has been possible to analyse two and three-dimensional structures of practical interest made of many stones of various size and disposition (until 1500 scalar unknowns), evaluating the collapse loads and the collapse mechanisms both in the presence of increasing horizontal body forces that simulate statically the seismic action and in the presence of other load conditions. In all the cases examined both the rotational and the sliding mechanisms at the interfaces between blocks are taken into account.
In all the cases here studied the load condition is the self- weight (dead load) and horizontal body forces (live load) that statically simulates the seismic actions.
In particular, Figure 1 shows the solution of the NLP problem obtained for a wall in 'opus africanum' starting from an arbitrary initial estimate (a) and from the initial estimate provided by the LP problem (b+c). It is easy to recognise that in the former case the NLP does not converges to the optimal solution, while in the latter case it converges in a short time. Note that the collapse mechanism strongly depends on the disposition of the piers, and in particular on the presence of the lintel beams.



$\alpha = 0.106$ $\alpha = 0.377$ $\alpha = 0.347$

Figure 1 Collapse mechanism and collapse multiplier for a wall in 'opus africanum' under inclined body actions: (a) NLP solution from arbitrary initial guess; (b) LP solution; (C) NLP solution from LP solution.

Figure 2 shows the results of the NLP problem, obtained for masonry walls with openings made of bricks of different size. These results are in a good agreement with experimental results (Baggio, Trovalusci 1993, 1995) and point out the influence of the bricks size on the collapse behaviour of walls.

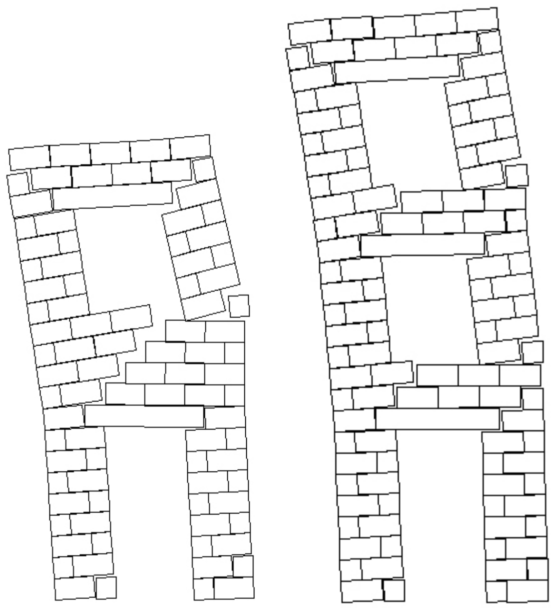


Figure 2 Walls with openings: collapse mechanism

As in the case of Figure 1, the collapse load multiplier, α^+, and the collapse mechanism, λ^+**,** resulting from the first step of the analysis (LP) often gives a good estimate of the final (NLP) multiplier and mechanism, α and λ**.** Therefore, **it** can be often useful to stop the analysis to the linear solution avoiding the burden of executing the non-linear step. Especially in a three-dimensional frame in which the number of the unknowns involved, even for systems of few blocks, is high.
Some results of limit analysis of three-dimensional structures made of stone in dry contact always subjected to the action of the self–weight (dead load) and horizontal out-of-plane body forces (live load) are reported in the following figures. All the cases studied, solving the LP problem, pointed out the influence of torsional mechanisms.

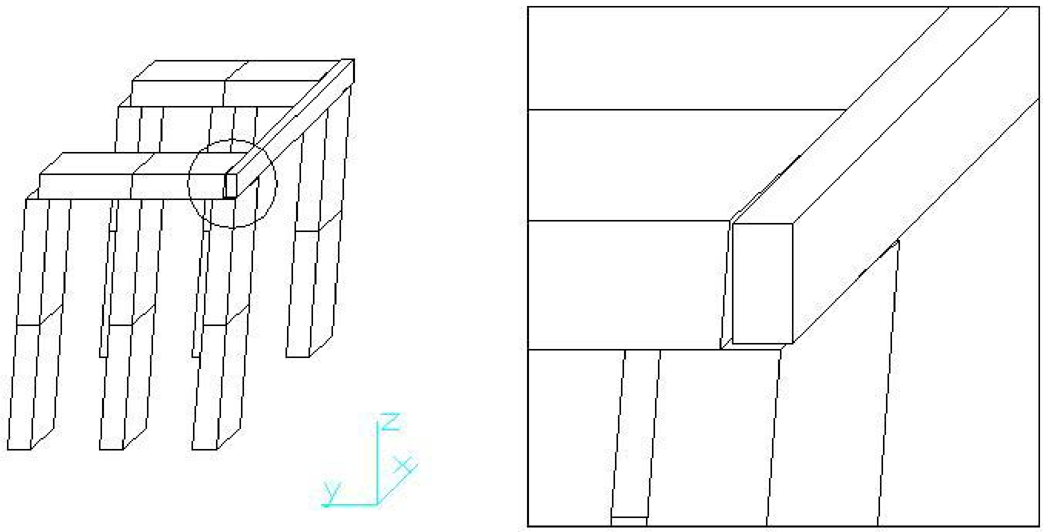


Figure 3 Trilithic structure: collapse mechanism

Figure 3 shows a trilithic structure made of 29 blocks and 38 joints that gives rise to a quite simple linearised problem: 38x10 kinematical unknowns plus α. Input file contains block masses and mass centres, joint orientations in space and a connection matrix numbering the two blocks in contact along any interface. The whole computer time sums up to 26 seconds.

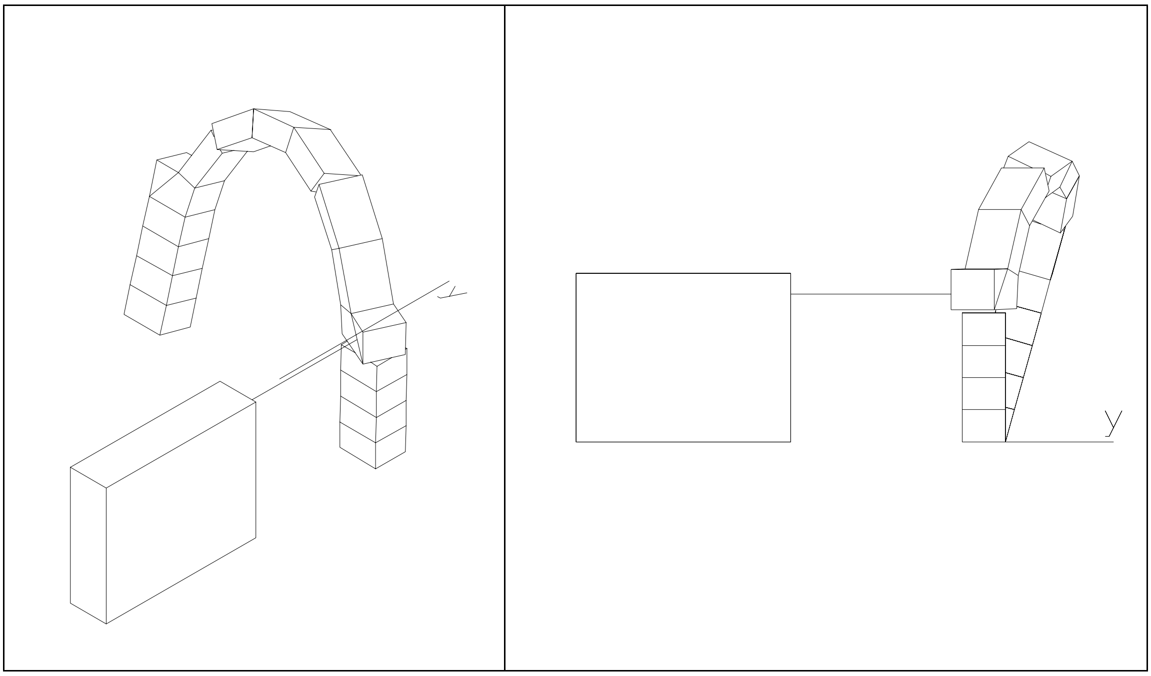


Figure 4 Vaulted structure with chain anchorage: collapse mechanism

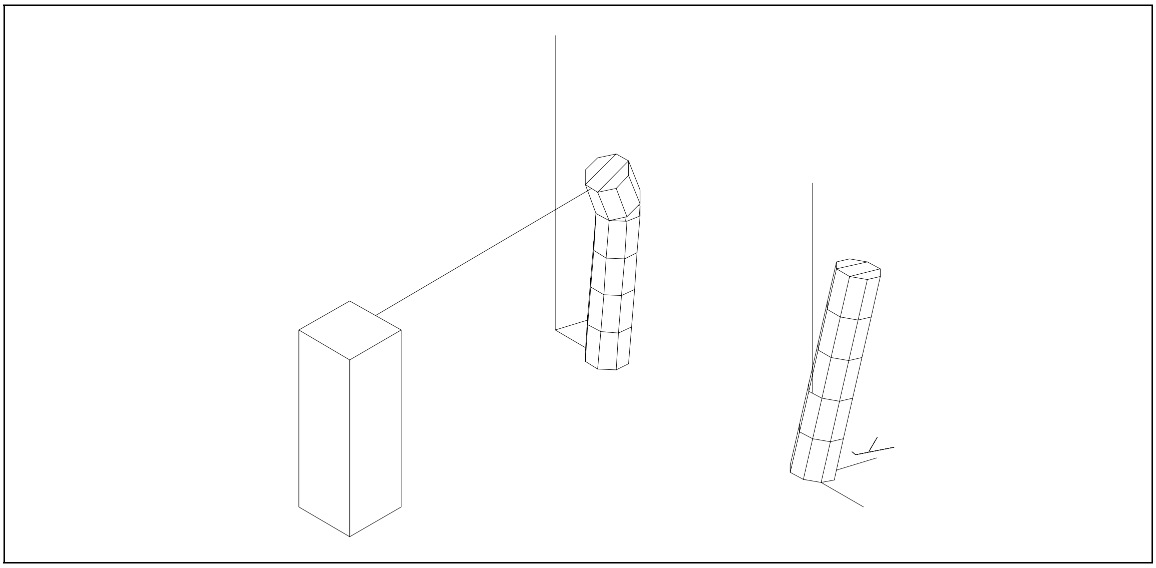


Figure 5 Comparison between collapse kinematisms with or without chains

A recent improvement of the model has been the introduction of 'tie-elements' (tension-only elements) that can represent the presence of a reinforcing bar between blocks (examples of Figures 4-6).

In Figure 4 an arch connected to a wall through a chain is studied. The limit analysis results show the activation of significant torsional mechanisms in the joints.
Figure 5 shows comparison between two 3D samples. The columns are made by a few drums, as actual ancient marble columns; a chain to a fixed element connects one of the two. The chain is modelled by a tension-only element, which controls relative movements between two points of two different bodies. Each octagonal joint in the model can undergo rotations around the 8 edges, it can slide along eight different directions and rotate along the normal axis. The whole run time, from reading input to writing output, sums up to three seconds. For the freestanding column collapse factor α is equal to 0.2 (base to height ratio), whereas for the column bonded on the top α comes out to be 0.5. Moreover the limit analysis clearly shows a significant modification of the collapse mechanism.

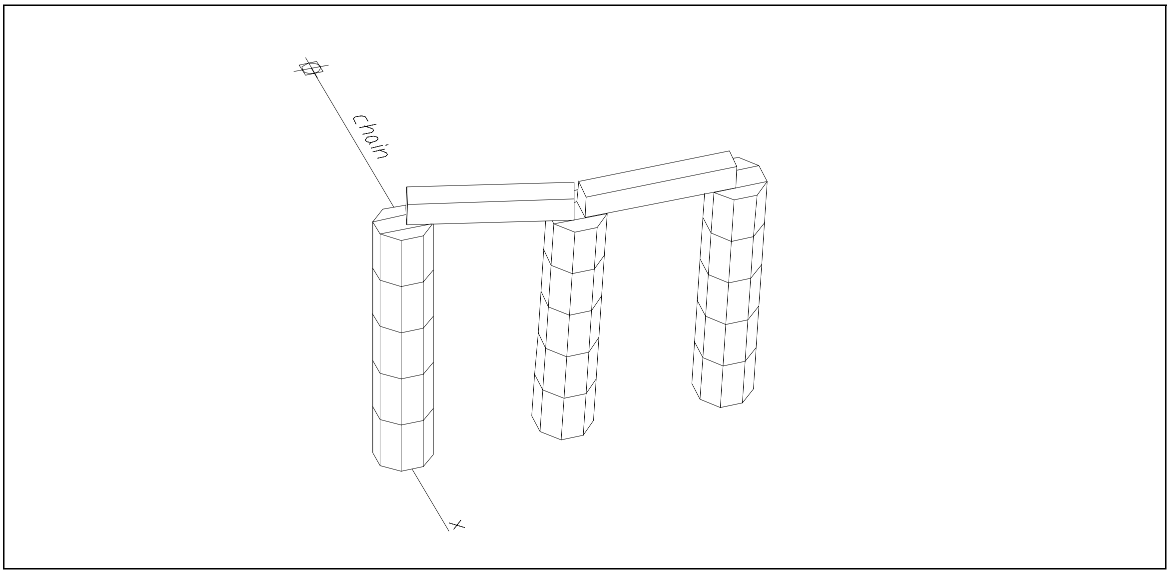


Figure 6 Columns system with constraints: collapse mechanism

In Figure 6 three columns connected by architraves are shown. One of the columns is bonded by a chain to a fixed point; the live horizontal body action is directed along x-axis; the free columns undergo rotation along the edge parallel to y-axis, whereas the bonded one does not move; the weight of the architrave is sufficient to prevent joint rotation along the column.

4 Finally

Aim of this work is to show the usefulness of a discrete approach in the structural analysis of ancient block-masonry constructions, for which it is crucial taking into account the shape, the size and the texture of the stones. In particular, the suitability and the convenience of non-standard limit analysis via non-linear mathematical programming is recognised; in fact it permits to correctly estimate collapse mechanisms and collapse loads, without uncertainties related to the definition of material parameters and allows to account for sliding kinematisms.
The two-step computational procedure proposed allows coping with non-linear mathematical programming for structures of practical interest, with many degrees of freedom, and provides a proper and simple computational tool to perform analyses.
The procedure is completely automatic and it is easy to be adopted by any user, it requires only the introduction of the geometry of the structure and of the coefficient of

friction. The program is going to be adapted to be distributed to structural engineers.

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