MECHANICAL BEHAVIOUR OF ARCHES AND VAULTS

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SUMMARY

Masonry, as a unilaterial material, is incapable of carrying tensile forces, and will crack in response to displacements imposed by the external environment (for example, settlement of foundations). The resulting deformations of a masonry structure are very much larger than any "elastic" strains; moreover, since the imposed external displacements may be unknown, a conventional analysis will yield little relevant information. However, an examination of possible states of equilibrium confirms that the stability of masonry resides in its geometry – the shape of the structure must be correct.

1 INTRODUCTION

Figure 1, from a 1934 drawing by Pol Abraham, shows typical cracks in a quadripartite masonry vault; an aisle vault is illustrated, but similar defects may be seen in the central high vaults of a major church. The displacements necessary to produce such cracking are large – the span of the vault in fig. 1 may have increased by 100 or 200 mm. Despite such movements, which are outside the experience of a modern engineer used to the design of steel or concrete frames, there may well be general agreement that the vault of fig. 1 is not in a dangerous state. On the contrary, a masonry structure is thought of as robust, able to withstand significant movements of foundations, to survive earthquakes and indeed bombardment in time of war.

However, a vague belief that the vault of fig. 1 is safe must be supported by a more exact assessment of its state. How does that
vault carry its loads? (The loads are in general due only to its own weight.) And why has the vault cracked in precisely the way sketched? (The cracking is typical of most vaults.) Some sort of structural analysis is in fact needed to establish the mechanics of the vault. Without such a basic understanding repair work may be directed to wrong ends, or indeed may be harmful rather than beneficial.

The tools available to the structural engineer are well known. Only three types of equation may be written, and from these the structural quantities may be calculated. First, equilibrium must be satisfied – the external forces acting on a structure must be balanced by the internal stress resultants. Certain assumptions and approximations must be made in formulating these equations of equilibrium, but in fact the equations are unambiguous, and may be trusted. Secondly, material properties are introduced into the analysis, and here there may be large areas of uncertainty. The elastic and plastic properties of steel may be well known – they are less certain for reinforced concrete. For masonry, which is an assemblage of stone (in itself inhomogeneous) and mortar of variable or unknown properties, the material parameters in tension and compression can be very different – an assemblage of masonry elements may be unable to accept tensile stress, for example, and this must be allowed for somehow in the analysis.

Finally, the geometrical equations, the so-called equations of compatibility, may be almost impossible to formulate. These equations relate the internal deformations of a structure to the external displacements, and are exemplified by "boundary conditions". Knowledge of these displacement boundary conditions is essential for the elastic (Navier) solution of a hyperstatic structure, but it is a fact that very small changes in these geometrical constraints – the precise span of an arch, for example – will cause very large changes in the internal state of the structure. Disregarding this for the moment, an elastic solution for a typical masonry structure will indicate first, that the stresses are very low compared with the basic strength of the material, and second, that the deflections may well be markedly less than say 10 mm, compared with the movement of 100 or 200 mm that may be imposed on a masonry vault by displacement of its supports.

Fortunately, significant statements about the behaviour of masonry structures may be made on the basis solely of equilibrium, the most reliable of the three types of equation that can be used in structural analysis.

2 THE MATERIAL

Masonry made from stones, or bricks, or indeed of sun-dried mud (adobe, with or without a reinforcement of straw) has little tensile strength. On the other hand, as has been noted, compressive stresses are low, and the usual "engineering" assumptions for masonry are that

(a) it has no tensile strength,
(b) it has virtually infinite compressive strength, and
(c) slip does not occur between components of the structure.

With these assumptions, a structural analysis of masonry may be made by using only equations of equilibrium, and disregarding elastic distortions and conditions of compatibility. The analysis falls within the framework of plasticity theory, and powerful theorems may be invoked.

3 THE SIMPLE ARCH

The behaviour of the simplest possible masonry structure, the voussoir arch, shows how an analysis can be made. Figure 2(a) shows the arch; during construction the masonry must be supported on temporary falsework. Once this centering is removed, the
completed arch will require support from its abutments - the thrust of the arch will force the abutments to give way slightly, and the span of the arch will increase.

Figure 2(b) shows how the arch accommodates itself to the increased span - the figure is much exaggerated. The arch will crack at joints between voussoirs; there is no strength in these joints, and three hinges have formed (which in practice may be revealed by cracking of any mortar that may be present). The three-hinge arch is a well known and perfectly stable structure; there is no suggestion that the arch of fig. 2 is in a state of distress. On the contrary, it has merely responded in a sensible way to attack from a hostile environment. A first fundamental statement relating to the assessment of arches may be made on the basis of this simple example; cracks in masonry are not necessarily signs of distress. Cracks reflect the natural state of a stone structure; the secret of the behaviour of masonry is that it can adjust to randomly imposed external deformations by such harmless cracking.

The dotted line in fig. 2(b) indicates the way the forces are transmitted within the arch. Each of the voussoirs pushes against its neighbours; at the crown of the arch there is only a single point of contact, and the line of thrust must certainly pass through that point, as it must through the hinges at the abutments. The shape of the line of thrust in fig. 2(b) was identified by Robert Hooke in 1675 - it is that of the corresponding inverted weightless cord, carrying the same loads in tension (rather than the compression of the arch).

The dotted line in fig. 2(b) is located precisely, since it must pass through the three hinge points. However, if the arch is not subjected to the particular movements of the abutments implied by fig. 2, then the line of thrust - the inverted hanging cord - can not be drawn immediately. Figure 3 is based on a sketch illustrating a nineteenth-century examination of the equilibrium of arches; three lines of thrust are shown, of which the steepest corresponds to the dotted curve of fig. 2(b). The shallowest curve in fig. 3 records the state of the arch should the abutments approach rather than spread, as they might if the arch were an internal span of a multi-span bridge. Between these two extremes is shown a curve following roughly the centre line of the arch, and in fact an infinite number of such curves can be included within the thickness of the masonry.

Figure 3. Alternative lines of thrust
Very small, almost infinitesimal, movements of the abutments will cause the line of thrust to adopt one rather than another of the possible locations. However, these abutment movements, defining the geometrical boundary conditions for the arch, are not only unknown, but unknowable, depending as they do on changes in soil conditions, the passage of a heavy load, even wind loading from a gale, and so on. A second fundamental statement about the behaviour of masonry may be made on the basis of this example of the simple arch – there is no way of determining the "actual" equilibrium state of a hyperstatic structure.

However, the powerful "safe theorem" of plasticity theory may be used. For the masonry arch, if one state of equilibrium can be found for which the structure is stable, then it will be stable absolutely; it is not necessary to try to determine the "actual" state. Thus, in fig. 3, any one of the three inverted hanging cords represents a line of thrust lying wholly within the masonry, and is hence a demonstration that the arch is safe. Very small unknowable movements of the abutments may cause the line of thrust to move violently within the masonry, but it can never escape.

The arch must, of course, have the right shape. Thus a semicircular arch, for example, which is able to carry its own weight, must have a minimum thickness (which turns out to be just over 10 per cent of the radius). A thinner arch will not stand: a thicker arch will always be able to contain a thrust line corresponding to the inverted hanging cord.

A third general fundamental statement may be made on the basis of these arguments. Stability of a masonry structure is assured by its shape – the geometry must be correct. In modern times mathematics may be used to verify the geometry, either by direct calculation, or on the drawing board, by the use of graphic statics. Before the use of mathematics, the structure itself confirmed stability – the fact that a masonry structure is seen to exist is a demonstration, without calculation, that a set of forces exists within the fabric which equilibrates the weight of the structure. Thereafter small movements of the environment may cause considerable changes in the equilibrium state, and they may cause cracking, but they can never of themselves cause collapse.

4 THE MASONRY VAULT

The defects in the quadripartite vault in fig. 1 may be interpreted by these simple ideas about the masonry structure – at the same time, as will be seen, a clear view is obtained of the way the forces are carried in the vault.

A first step lies in the examination of a simple barrel vault, fig. 4, which is nothing more than a three-dimensional translation of the arch of fig. 2. The cross-section in fig. 4 is drawn roughly to scale (say a vault with thickness 300 mm over a span of 12 m). The vault is maintained by external buttressing. The arch of the vault is too thin to carry its own weight right down to the springings, and rubble fill is shown backing the haunches so that the thrust (the inverted hanging cord) can escape from the vault proper. In fig. 4(b) the buttressing system has given way slightly, and the crack pattern of
Fig. 2(b) has developed. The hinges in the extrados near the fill will not be seen from below, but the more or less central hinge will be visible.

The cross-section of the vault in fig. 4 is the same down the length of the barrel. Figure 5 shows a single bay of a quadripartite vault formed by the intersection of two slightly pointed barrels. If now the buttressing of the vault gives way, the portion running east-west will crack as before, and the single hinge line at or near the crown will be seen from within the church. The change in geometry caused by the increase in span will result in a drop of the crown of the vault.

There is a severe geometrical mismatch in the intersecting barrel which runs north-south - there is not enough masonry to fill the increased north-south dimension. A crack pattern which allows the vault to deform in virtually strain-free monolithic pieces is sketched in figs 5(b) and (c). Sabouret cracks (named after the architect who made an analysis of vaults in 1928) have opened in the vault at a distance of a metre or so from the north and south walls (usually containing windows). In addition, cracks may have opened adjacent to the walls. These cracks in the north-south barrel represent complete separation of the masonry, and both types of crack - hinging and separation - may be seen in the sketch of fig. 1.

Once again, such crack patterns do not indicate that the vault is on the point of collapse. Rather, they give more or less precise indications of response to externally imposed movements. Moreover, it is clear that the edges of the vault are acting as simple arches, as is indicated schematically in fig. 6.
5 CONCLUSION

Masonry is a material which, in itself, is virtually rigid and infinitely strong in compression. It cannot, however, accept tension, and will crack in response to attempts to impose tensile forces. Any distortions which may be visible in a masonry structure are the results of the development of such cracks:

The cracked state is the natural state of masonry.

Cracks indicate that the environment has imposed external deformations on the structure – movements of the foundations, for example. Very small movements of this sort can lead to very large changes in the equilibrium state of the structure – the way in which the loads are carried. Since such movements are unpredictable, the calculation of the “actual” state of a structure cannot be made:

There is no unique calculable equilibrium state for masonry.

Thus an elastic analysis of a masonry structure, perhaps by using finite elements, will certainly generate an equilibrium solution. However, the elastic analysis relies on known boundary conditions, and since these conditions are essentially unknowable, the solution is one which will not be observable in practice. Further, elastic displacements for masonry are very small, and negligible in comparison with distortions imposed by the environment; no significant information will result from an elastic deflexion analysis.

Instead, reliance may be placed on the “safe” theorem of plasticity theory. If any one equilibrium solution can be shown to be satisfactory – the forces lie within the surfaces of the masonry – then this guarantees the safety of the structure. This is a statement about geometry:

The correct shape of a masonry structure ensures its safety.

6 BIBLIOGRAPHY

