Transverse cracking and horizontal thrust in flattened arches

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ABSTRACT: In flattened stone and concrete arches, where the dead load is decisive, the greatest bending moments occur at the abutments. These bending moments induce opening of dry joints and cracks with gaps, which change considerably the equilibrium state of the arch. This change can be reinterpreted by linear transformations as a decomposition of the structure into two components: an uncracked monolithic elastic structure and an assemblage of rigid blocks with partial interpenetrations. This decomposition provides the tools for the determination of the horizontal thrust of the arch. Accordingly the effect of the rise and the slenderness of the arch on the horizontal thrust corresponding to dry joints at the springings has been analysed. Special attention is paid to the straight arch with dry joints and ditto brittle cracks at the supports. Subsequently the interrelation of the effect of dry and brittle behaviour on the horizontal thrust has been evaluated.

1. INTRODUCTION

The volumes V_h of the voids caused by transversal cracks and dry joints induce in the arch elongations Δl_h of its centroidal axis and mutual rotations ω_h of the endfaces at the abutments. If these movements are restrained they give rise to an increased horizontal thrust H that attains its highest value H_{max} when the abutments are fixed. Because this H determines the design of the support structure of the arch we focus our attention on the overall effect of the cracks and dry joints on H. We consider purely elastic arches with possible nondissipative effects at the joints. Because the greatest bending moments, also after cracking, occur at the abutments we restrict ourselves to flattened arches with cracks only at the springings. We confine ourselves to symmetrically loaded symmetric arches with constant rectangular cross-section A = td supported by fixed abutments.

2. THE EFFECT OF TRANSVERSAL CRACKS ON THE ARCHES

We consider a straight crack (i) of depth a, characterised by a displacement discontinuity or gap vector $\mathbf{u}_{i+1}(y) - \mathbf{u}_i(y) = \mathbf{?}(y)$ in a local coordinate system s, y(s) (Fig. 1). The gap vector **?** with longitudinal component γ_s and transverse component γ_y

$$\mathbf{?}(y) = ?_s \mathbf{n}(s) + ?_y \mathbf{j}(s) \quad \text{where } ?_s \geq 0 \tag{1}$$

determines the non-negative gap volume V_h and its moment $y_h V_h$

$$V_h = \int ?_s dA \quad ; \quad y_h V_h = \int ?_s y dA \tag{2}$$

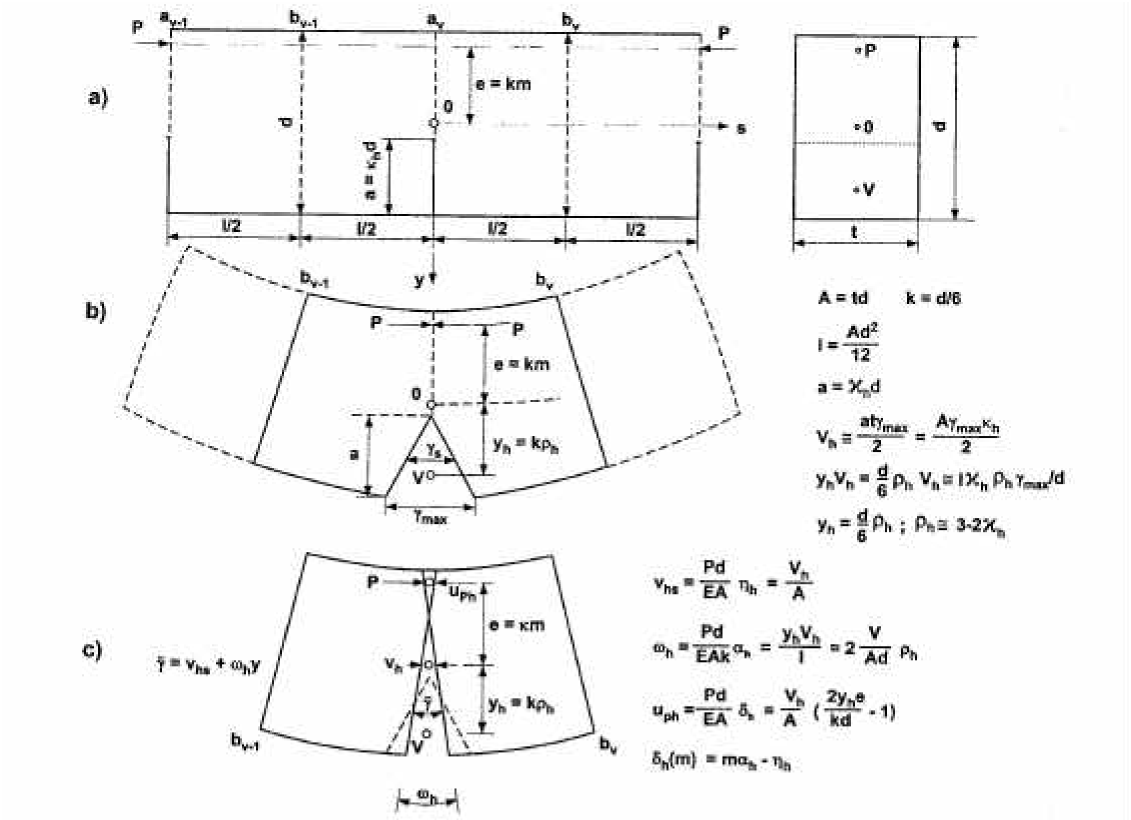

Figure: 1 a) Ashlars with transverse cracks. b) Deformed ashlar assemblage with opened symmetric crack at joint. c) Linearized symmetric crack

The elastic stress fields σ', σ", σ'" generated at (i) by normal force N = -P = 1, moment M = 1 and shear force Q = 1, respectively, have the non zero components

$$s'_s = 1/A \quad ; \quad s''_s = y/I \quad ; \quad t''' = \frac{3(1-(2y/d)^2)}{2A} \tag{3a,b,c}$$

Applying these stress fields to the void induced by **?** we obtain by work equations the constant discontinuities at (i) (Parland and Miettinen 1998)

$$\int s'_s\, ?_s dA = V_h/A = v_{hs} \; ; \int s''_s\, ?_s dA = y_h V_h/I = ?_h \; ; \int t'''\, ?_y dA = v_{hy} \tag{4a,b,c}$$

Together these equations define a linearized gap distribution at (i)

$$? = \begin{Bmatrix} ?_s \\ ?_y \end{Bmatrix} = \begin{Bmatrix} v_{hs} + ?_h y \\ v_{hy} \end{Bmatrix} \tag{5}$$

The equation expresses that at joints (i) contacting endfaces experience mutual translations v_{hs}, v_{hy} and rotations ω_h. Therefore we interprete **?** as a mutual rigid body movement between coherently deformed endfaces of blocks of a monolithic arch. These blocks deform according to the classical theory of bending and compression of beams with strains

$$e_{se} = -\frac{P(s)}{EA} + \frac{M(s)y}{EI} \; ; e_{ye} \cong -?e_{se} \; ; ?_{sye} = \frac{3Q(s)}{2GA}(1-(2y/d)^2) \tag{6a,b,c}$$

induced by the compressive normal force P, bending moment M and shear force Q, respectively. Therefore it is possible to decompose the state {σ , ε , u} of the arch into a monolithic elastic part {σ_e , ε_e , u_e} with continuous displacement field and a quasi-rigid part {σ_h , γ , u_h} of rigid blocks, that at the cracks experience mutual translations v_{hs} , v_{hy} and rotations ω_h , including partial interpenetration at the compressed edge (Fig. 2)

$$\{s,e,u\} = \{s_e, e_e, u_e\} + \{s_h, ?, u_h\} \tag{7}$$

The states $\{\sigma_e , \varepsilon_e , u_e\}$ and $\{\sigma_h , \gamma , u_h\}$ are inter-related by the conditions at the fixed supports

$$u(\pm L/2) = u_e(\pm L/2) + u_h(\pm L/2) = 0 \tag{8}$$

The stress energy of the monolithic component is

$$W_e = \frac{1}{2}\left(\int_{-L/2}^{L/2} \left(\frac{P^2}{EA} + \frac{5Q^2}{6GA} + \frac{M^2}{EI} \right) ds \right) \tag{9}$$

The state $\{\sigma_h , \gamma , u_h\}$ is determined by the linearized discontinuity components v_{hs} , v_{hy} , ω_h at the joints. The compressive force P at joint (i) acts at y_p = -e. There the rectified "ashlars" interpenetrate each other a distance (Fig. 1)

$$u_{ph} = -v_{hs} + ?_h e = \frac{V_h}{A}\left(\frac{12 y_h e}{d^2} - 1 \right) \tag{10}$$

We express v_{hs} , v_{hy} , ω_h , u_{ph} of a crack and its centroid ordinate y_h , respectively, by

$$v_{hs} = \frac{Pd}{EA} ?_h \; ; \; v_{hy} = \frac{Pd}{EA} e_h \; ; \; ?_h = \frac{6P}{EA} a_h \; ; \; u_{ph} = \frac{Pd}{EA} d_h \; ; \; y_h = \frac{d}{6} ?_h \tag{11}$$

where the parameters η_h , ε_h , α_h , δ_h and ρ_h are functions of P and its relative eccentricity m = 6e/d, the shear force Q = qP and the relative crack depth κ_h = a/d. Because of Eqs. (10) and (11) there holds

$$d_h = m a_h - ?_h \quad ; \quad ?_h = \frac{3 a_h}{?_h} \tag{12a,b}$$

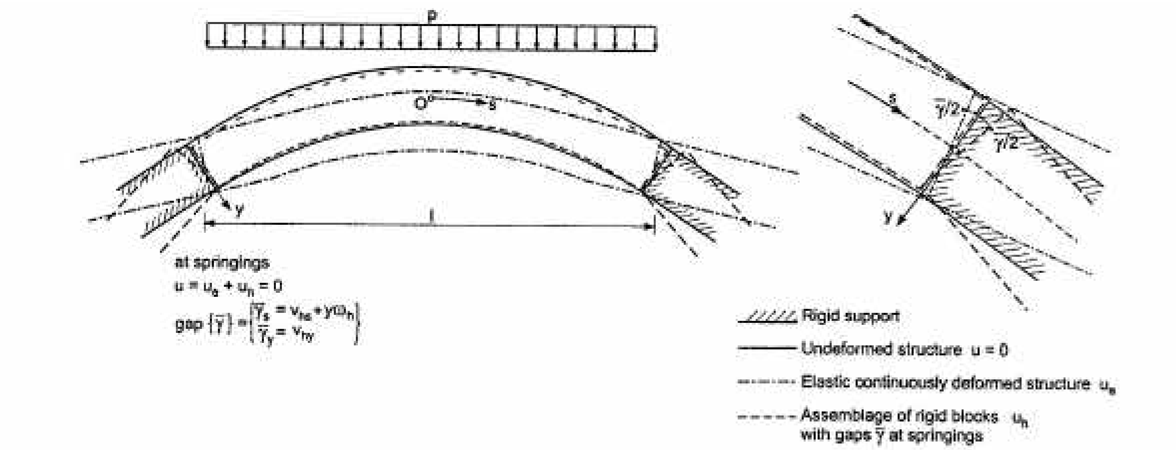

Figure2 : decomposition of deformations of elastic arch into a monolithic elastic component and a quasi rigid component with partial interpenetrations

The geometric compatibility implies that v_{hs} and ω_h determine also the shortening of the centroid axis and the mutual rotation of the endfaces of the ashlar, respectively. For symmetric loading upper bound values of the parameters η_h , ε_h , ω_h , δ_h have been determined by admissible equilibrium stress fields (Parland et al. 1982) and lower bound values by the FEM and ditto values for nonsymmetric loading by Miettinen (1988).
The stress energy of the edge effect at (i) after cracking is

$$W_{hi} = \frac{P^2 d}{2EA}(d_h + qe_h) \tag{13}$$

and the corresponding total stress energy of the arch is

$$W = W_e + SW_{hi} \tag{14}$$

If the arch and the load are symmetric we have two redundants, the horizontal thrust H and the bending moment X at the springings. Using the elastic center of the arch as the origin (Charlton 1969) with vertical coordinate z(s) and inclination υ(s) of the centroid axis of the arch, the P(s), Q(s) and M(s) are then determined by

$$P(s) = P^o(s) + H\cos? \; ; \; Q(s) = Q^o(s) - H\sin? \; ; \; M(s) = M^o(s) - H\,z(s) + X \tag{15}$$

where $P^o(s)$, $Q^o(s)$ and $M^o(s)$ represent the corresponding forces of the statically determinate structure. Because of the smallness of the strain γ_{sy} within the ashlars and the gap vector component γ_y at the joints, these quantities are disregarded in the following.

3. THE FLATTENED MONOLITHIC ARCH WITH DRY JOINTS AT THE SPRINGINGS

We base our analysis on the general results obtained for elastic voussoir arches. Provided the slenderness of the voussoirs $\lambda = l/d$ exceeds the value 1.5, the edge effects caused by the opened cracks are independent of λ (Parland et al. 1982).

At dry joints, because of unconstrained contact without stress singularities, the opening gaps γ_s attain their maximum depth a^o and the void-load lever arm $VP = e + y_h = k(m + \rho_h^o)$ attains a minimum (Figs. 1 and 6)

$$v_{hs} = \frac{Pd}{EA}?_h^o(m)\; ; \; ?_h = \frac{Pd}{EAk}a_h^o(m)\; ; \; u_{ph} = \frac{Pd}{EA}d_h^o(m)\; ; \;\; (k = d/6) \tag{16a,b,c}$$

The condition of fixed abutments corresponds to a minimum of stress energy $W = W_e + W_h$, that provides the necessary equations for the redundants H and X (Parland and Miettinen 1998)

$$H\int_L \left(\frac{1}{EA}\left(\cos^2 ? + (z/l)^2\right)\right) ds = \int_L \left(\frac{N^o \cos?}{EA} + \frac{M^o z}{EI}\right) ds + \frac{Pd}{EA}\left(?_h^o \cos? + (z/k) a_h^o\right) \tag{17a}$$

$$X\int_L \frac{ds}{EI} = -\int_L \frac{M^o}{EI} ds - \frac{Pd}{EAk} a_h^o \tag{17b}$$

The first terms to the right represent the monolithic elastic component of the structure without cracks and the second terms the effect of the discontinuities induced by the movements of the rigid block assemblage of arch and abutments (Fig. 2).

The linear arch that passes through the intrados at the springings and the extrados at the crown determines at given load the smallest horizontal thrust H_{min}

$$H_{min} = M_{max}^o/(d + f) \tag{18}$$

Fig. 3 shows the dependence of the ratio H/H_{min} of flattened arches on the slenderness ratio $\lambda = l/d$ and the rise ratio $\theta = f/d$. For small slendernesses ($\lambda < 10$) H/H_{min} decreases steeply approaching thereafter asymptotically a lowest value.

An exception makes the straight arch for which $H/H_{min} \approx 1$ and therefore almost independent of λ. At any symmetric load with $z = 0$, $s = x$, $N^o = 0$, $\cos\upsilon = 1$, $X = -Pkm$, $\rho_h^o = 3\alpha_h^o/\beta_h^o$, we obtain from Eqs. (17a,b)

$$? = l/d = ?_h^o(m) \quad ; \quad H = 6\frac{\int M^o dx}{(m + ?_h^o)dl} \tag{19a,b}$$

For uniform load p the thrust H is nearly proportional to λ (Fig. 7)

$$\frac{H}{pl} \cong \frac{?}{2(m + ?_h^o)} \tag{20}$$

If m > 2 the sum $m + \rho_h^o$, corresponding to different loads, doesn't differ more than 3% from the value 4. Therefore the gap volume lever arm VP = $e + y_h$ is almost constant and nearly independent of Q (Miettinen 1988). For a rectangular section there holds (Fig. 4)

$$e + y_h \cong 2d/3 \quad \text{or} \quad m + ?_h^o \cong 4 \tag{21a,b}$$

Thus

$$\frac{H}{pl} \cong \min\frac{H}{pl} = \frac{?}{8} \tag{22}$$

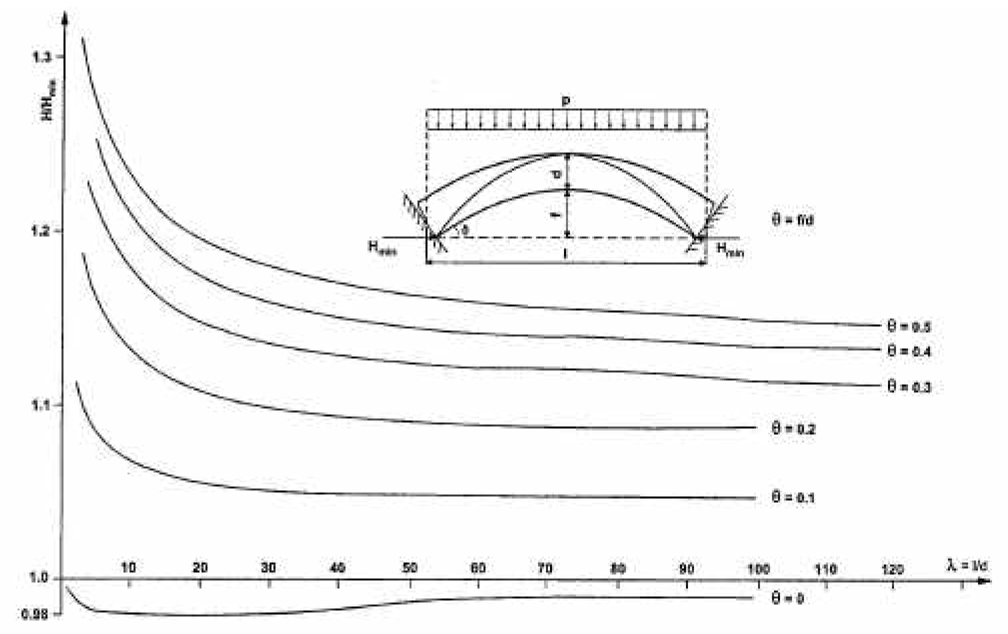

Figure 3 : Dependence of ratio of horizontal thrust H to H_{min} on slenderness ratio λ and rise ratio θ = f/d of an arch with dry joints at the sprinings

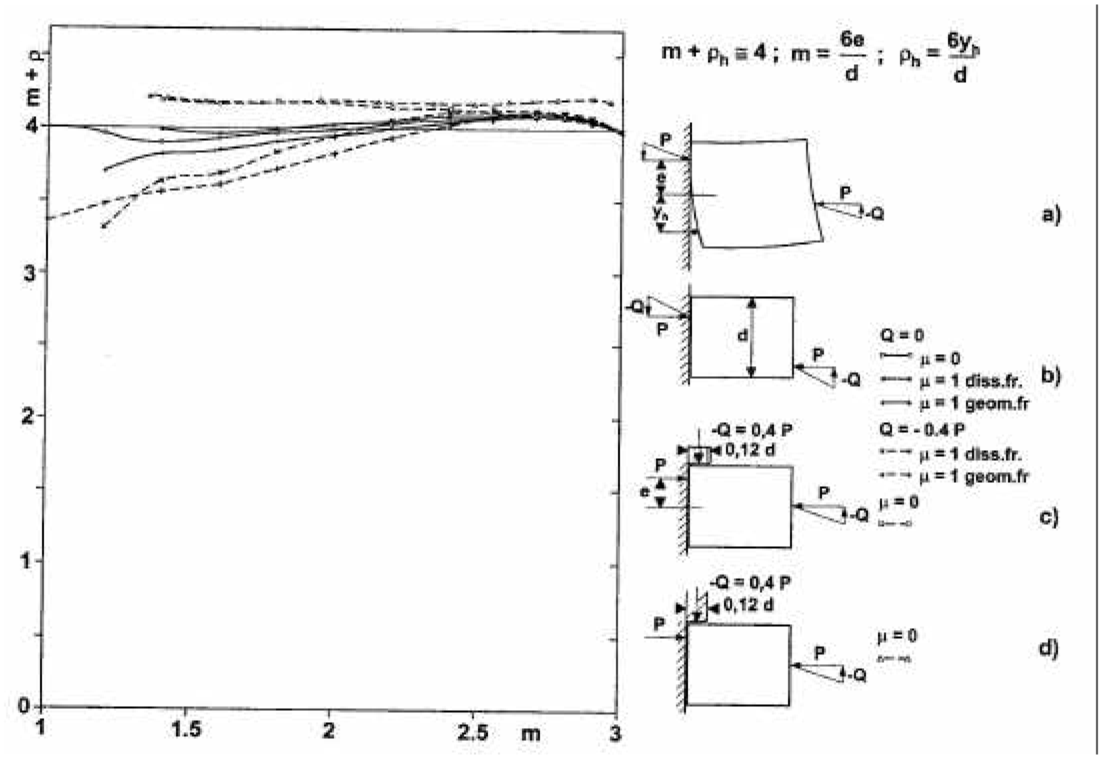

Figure 4 : Dependence of the internal lever arm ratio VP/k = m + ρ_h^o(determined by the centroid V of the void and the centroid P of the normal stresses) on different load and support conditions at dry joint.

The secondary effect of the deflection u_y on the bending moment $\Delta M = Hu_y$ together with the shortening of the span $½\int(u_y')^2dx$ decreases the growth of the horizontal thrust H with λ and leads finally to the snap-through of the straight arch (Fig. 5).

4. THE STRAIGHT ARCH WITH BRITTLE CRACKS AT FIXED SUPPORTS

We consider a clamped straight arch between two rigid supports loaded uniformly by p. This we interprete as a continuous beam with equal spans l loaded by uniform load p which induces solely normal stresses σ_x (friction neglected; $\mu = 0$) in the cracked sections above the supports (Figs. 4d and 6). Before cracking the state of stress is determined by the bending moment M(x)

$$M(x) = M^o(x) + M_s \; ; M^o(x) = px(l - x)/2 \; ; M_s = -2M^o_{max}/3 = -pl^2/12 \qquad (23a)$$

Neglecting the effect of the shearforce the potential energy U_o equals the negative strain energy induced by M(x). Hence

$$U_o = -\frac{1}{2}\int_0^l \frac{M^2}{EI}dx = -\frac{(pl)^2 d?^3}{120EI} \qquad (23b)$$

A crack at the supports with depth $a = \kappa_h d$ induces in the beam a constant horizontal thrust P = H acting at an eccentricity e = -km and a bending moment M_p = -Pkm. The edge effect on the cracks at the supports is in this case determined by the relations (11) and (12) where the parameters η_h , α_h , δ_h , ρ_h depend on the relative eccentricity m and the relative crack depth κ_h

$$?_h = ?_h(m, ?_h) \; ; a_h = a_h(m, ?_h) \; ; d_h(m, ?_h) = ma_h(m, ?_h) - ?_h(m, ?_h) \qquad (24)$$

The potential energy U in the cracked state equals the negative sum of the stress energy of the monolithic beam and of the edge effect (Eqs. 9 and 13)

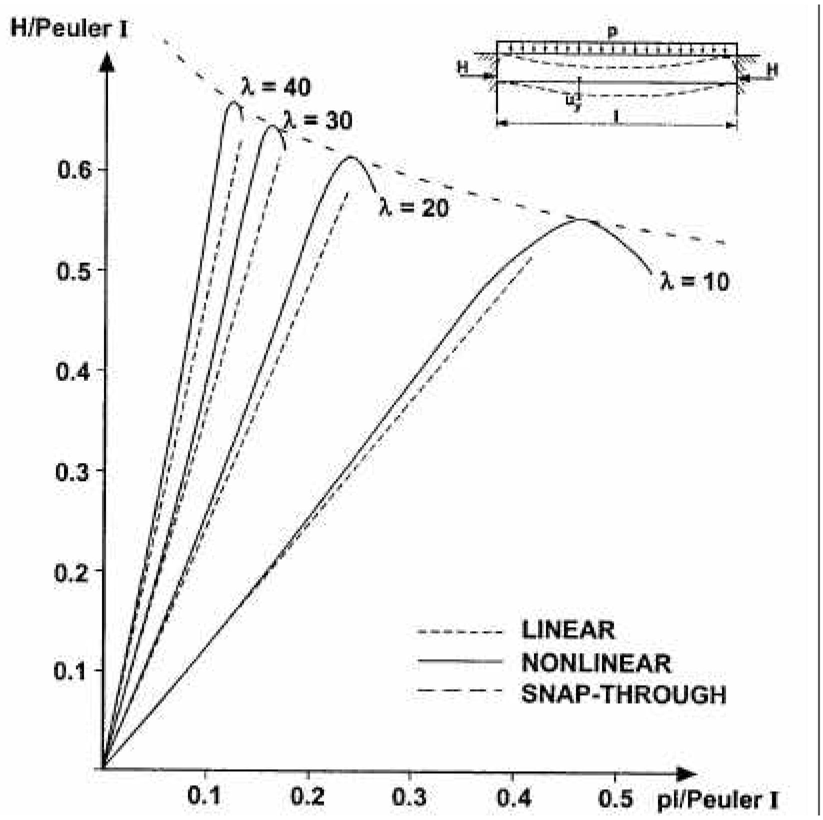

Figure 5 : Nonlinear effect of deflection and snap-through of loaded straight arch with dry joints at the supports.

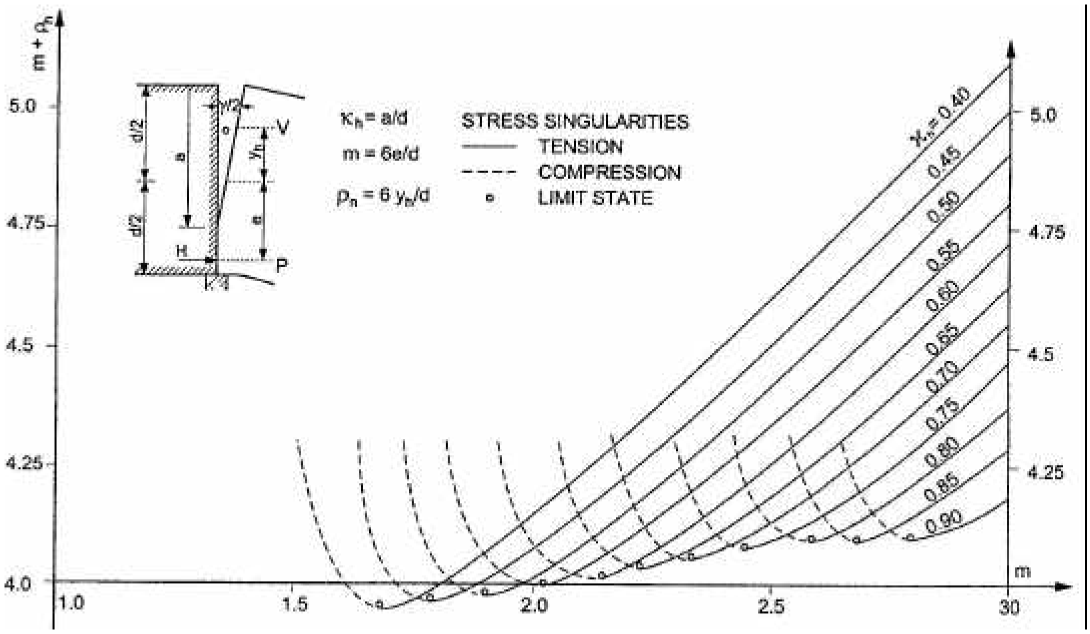

Figure 6 : Lever arm ratio VP/k = m + ρh versus relative eccentricity m at cracked support.

$$U = -\frac{1}{2}\left(\int_0^l \left(\frac{H^2}{EA} + \frac{(M^o(x) - Pkm)^2}{EI}\right) dx + \frac{P^2 d}{EA} d_h(m, ?_h)\right) \qquad (25a)$$

The change of the potential energy is therefore

$$? U = U - U_o = -\frac{d}{2EA}\left(H^2\left(?(1+m^2/3) + d_h(m,?_h)\right) - \frac{Hplm?^2}{3} + \frac{(pl)^2}{12}\right) \qquad (25b)$$

Using the linearization of the gaps $? = v_{hs} + ?_h y$ all relations concerning v_{hs}, ω_h, u_{ph} (Eqs. 16 and 17) remain formally in force. Therefore

$$? = ?_h(m,?_h) \quad ; \quad H = \frac{pl}{2(m+?_h)} \qquad (26)$$

Inserting these values into Eq. (25b) we obtain finally

$$? U = -\frac{(pl)^2 d}{24EA}\left(\frac{?_h^3 ?_h}{m+?_h}\right) \qquad (27a)$$

The internal lever arm ratio VP/k = m + ρ_h depends almost linearly on m at given crack depth ratio $\kappa = a/d$ (Fig. 6). With surface energy Ga the total energy change is

$$p = Ga - \frac{(pl)^2 d}{24EA}\left(\frac{?_h^3 ?_h}{m+?_h}\right) \qquad (27b)$$

According to linear fracture mechanics cracking requires (Hellan 1985)

$$\frac{\partial p}{\partial a} = G - \frac{(pl)^2 d}{24EA}\frac{\partial}{\partial ?_h}\left(\frac{?_h^3 ?_h}{m+?_h}\right) = 0 \qquad (28)$$

The condition for cracking depends on the form of the crack. For simplicity we choose an intermediate triangular $\gamma(y)$ distribution (Fig. 1)

$$?_s(y) = 2?_{max}(y - y_o)/d \geq 0 \qquad (29)$$

with corresponding

$$v_{hs} = \frac{V_h}{A} = \frac{?_h ?_{max}}{2} \quad ; \quad ?_h = \frac{y_h V_h}{I} = \frac{?_h ?_h ?_{max}}{d} \quad ; \quad ?_h = 3 - 2?_h \qquad (30)$$

Inserting ρ_h, $\eta_h = \lambda$ and m + ρ_h = pld/(EAγ_{max}) into Eqs. (27b) and (28) we obtain

$$p = Ga - \frac{pl?}{24}?_h(3-2?_h)?_{max} \; ; \quad \frac{\partial p}{\partial a} = G - \frac{pl?}{24}?_{max}(3-4?_h) = 0 \qquad (31a,b)$$

Hence

$$?_h = \frac{3}{4} - \frac{6Gd}{pl? ?_{max}} > 0 \qquad (32)$$

Because $0 \leq \kappa_h \leq$ ¾ the greatest value of the crack volume is $V_{h\,max} = 3A\gamma_{max}/8$. From Eq. (30) thus follows the minimum condition for a crack at λ and the load pl

$$? \geq \frac{3GAd}{pl\,V_{h\,max}} \qquad (33)$$

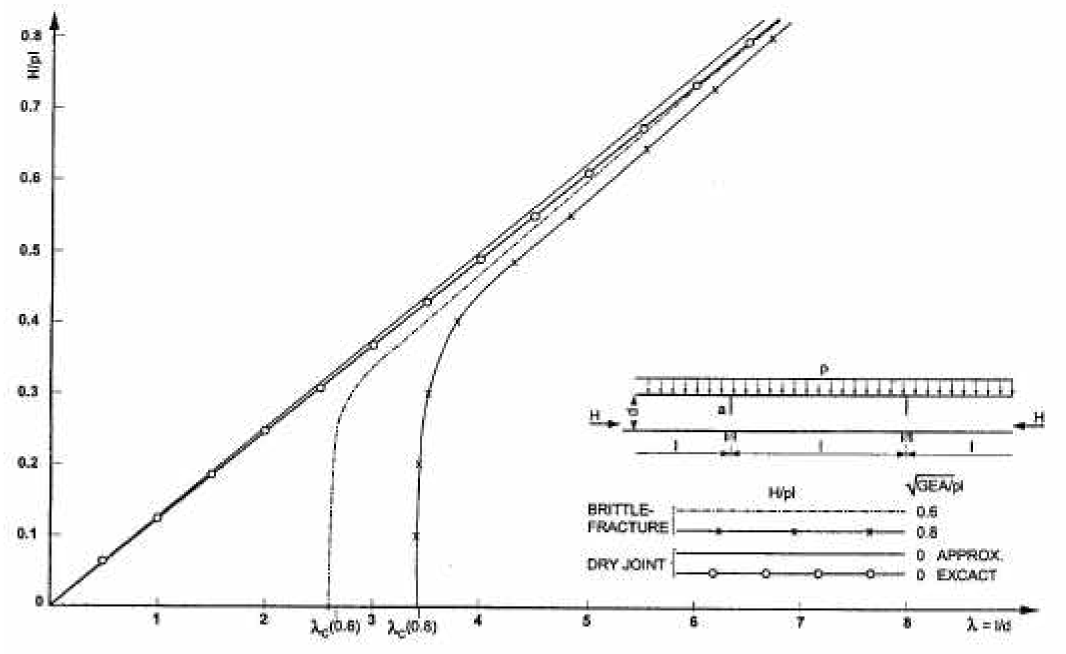

Figure 7 : horizontal thrust ratio H/pl versus slenderness ratio λ in the case of dry joints and in the case of brittle fracture.

Using Eqs. (31) and (33) we obtain approximate values for the dependence of the ratio H/pl on λ ($\lambda < 20$). The corresponding curves of (H/pl) for different ratios $\sqrt{GEA/Pl}$ have the asymptote $(H/pl)_o$ corresponding to dry joints at the supports (Fig. 7).

5. CONCLUSIONS

At uniform vertical loading p the greatest bending moments, also after cracking, occur at the springings. Therefore we have confined ourselves to arches and beams with cracks at the abutments only.

If the arch has dry joints at the springings then the ratio of the horizontal thrust H to its minimum value H_{min} increases with increasing ratio f/d, where f is the rise of the arch and H_{min} is the smallest horizontal thrust, to which corresponds a thrust line within the arch (Fig. 3).

The ratio H/H_{min} decreases with increasing slenderness λ but approaches a minimum constant value ($\cong 1$) when the rise f approaches zero (straight arch).

In the straight arch the ratio of the horizontal thrust to the total vertical load increases proportionally to λ.

A nonlinear analysis of the straight arch provides the conditions for the occurrence of a snap-through of the arch.

If the straight arch has tension resisting supports, cracks develope there at some critical slenderness ratio λ_c that increases with the toughness ratio $\sqrt{GEA/pl}$. For values $\lambda \geq \lambda_c$ the ratio $(H/pl)_G$ increases from zero discontinuously with λ and approaches asymptotically the value $(H/pl)_o$ corresponding to dry joints.

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