An evaluation of methods for the determination of the structural stability of historic masonry arches

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ABSTRACT: This paper presents a review and evaluation of the methods of analysis currently used for the determination of stability conditions for historical buildings considering the elastic and inelastic properties of the masonry. The most relevant mathematical models used for the simulation behaviour of the material are presented and compared. The relationship between the variation of the location of the thrust line obtained from the methods of analysis and the thermodynamics of irreversible processes is discussed within the framework of the results obtained from a masonry arch used as an illustrative example.

1 INTRODUCTION

The structural analysis of historic buildings with masonry vault roofs has been traditionally done using graphical static methods based on the determination of the thrust line. Recently, to solve the same problem linear elastic finite element techniques have been used frequently due to their potential to model structures of complex geometry, as is the case of colonial churches and other historic buildings. Considering that the geometry of these structures was originally conceived in such a way so that under the assumed loading conditions all their elements would be under compression, the results from this type of analyses closely describe their behaviour under the most common vertical loading conditions. The reason for this is that under these loading conditions, the assumption of linear elastic behaviour is reasonably correct. However, if the structure is subjected to other types of loading and/or lateral movements such as those induced by differential settlements or earthquake excitation, tensile stresses and associated cracking occur. This alters, in a significant manner, the stress magnitude and distribution in the structural elements and therefore modifies its stability characteristics. Unfortunately, these effects cannot be reproduced by the methods of linear analysis mentioned above.

Generally speaking, the results from linear elastic analyses of arch structures subjected to vertical loads are acceptable for the determination of the location of the thrust lines, because the energy liberation rate that occurs when the damage increases is relatively low. Unfortunately, this does not happen when the arch is subjected to lateral loads or support displacements as, under these conditions, non-linear effects due to damage cause the energy liberation rate to grow more rapidly invalidating the assumption of linear elastic behaviour. Thus, in order to consider this type of situation it is necessary to use methods of analysis to compute the location of the thrust line in the arch taking into account the non-linear behaviour of the material under tension.

Due to non-linear effects, the masonry presents cracked zones induced by tensile stresses that correspond to the behaviour observed in quasi-brittle materials, i.e., a reduction of the stress magnitude with increasing strain. This reduction of stress is a consequence of the damage produced in the structure prior to collapse. To model this type of behaviour for the masonry it is necessary to consider suitable constitutive models that involve internal variables to describe the damage evolution in the material all within the framework of the thermodynamics of irreversible processes. In others words, for the calculation of the stresses, it is important to consider the real behaviour of the material when subjected to strain states likely to occur in the real structure.

This paper presents a comparison between the linear and non-linear methods of analysis with different behaviour models for the masonry. For the non-linear models, the constitutive relations to model the inelastic behaviour of the masonry are based on the theory of plasticity, the theory of continuum damage and the concepts of smeared cracking for brittle materials. The damage evolution defined from the results of non-linear analyses allows for the observation of the corresponding variations in the thrust line and its relative location within the limits of the arch geometry and for the determination the sensitivity of the equilibrium states. It also presents the evolution that is of the damage energy of interest as a measure of the rate of entropy evolution of the system. The discussion of the final results is within the context of the thermodynamics of irreversible processes. The results of the non-linear analyses are evaluated particularly with special regard to the relationship between the rate of change of the damage energy and the location of the thrust line as a stability criterion. Finally, based on the results obtained from the illustrative example, some conclusions are drawn and recommendations for future work are given.

2 ARCH STABILITY

The importance of the use of collapse mechanisms in the study of masonry arch stability has been recognized since the early XVIII century. In those days the collapse analysis was carried out by introducing hinges and using equilibrium conditions to locate the thrust line and to verify that it laid within the geometric limits of the arch (Heyman, 1968). This approach was solely based on geometric criteria and ignored the real inelastic behaviour of the material when damage appeared. Considering that damage in the material produces noticeable changes in the position of the thrust line within the arch, it is evident that such a method did not represent the real behaviour of the structures. This made obvious the necessity to use more elaborate methods of analysis that consider the inelastic behaviour of the masonry during the deformation process.

In this paper we consider that the arch, in equilibrium between internal and external forces, is in a stable condition, if, by introducing small variations in load conditions, the location of the thrust line does not significantly change and still remains within the arch limits. This variation, in a numerical context, depends on the inelastic properties of the material, the geometric conditions and the application rate of the loads.

It is generally accepted that the local or material stability, has a strong influence on the global stability of a structure. Furthermore, the internal friction and the strain localization that the material experiences reduce its mechanical properties producing the softening behaviour of the masonry. This phenomenon causes the redistribution of stresses within the structure due to unloading in certain parts of the structure, and changes the position of the thrust line, but not necessarily out of the arch geometry.

During the non-linear analysis of an arch, an incorrectly specified convergence criterion, or when the structure is reaching its load carrying capacity, a sudden strain a spurious softening on the material may force the thrust line out of the arch limits, (Bazant and Planas, 1998). Hence, during the analysis, it is necessary to verify the objectivity of the mesh or the step size of the loads (Roeder, 1998).

3 METHODS OF ANALYSIS FOR MASONRY STRUCTURES

Currently, most of the methods of analysis used for the study of historical buildings still consider masonry as a linear elastic material. However, it is widely known that this consideration is only valid up to a certain established limit, beyond which the material incursions into the inelastic range.

Among the most important linear and non-linear methods of analysis used for this type of problem are the following: force equilibrium based methods, elastic-energy methods and methods which use advance numerical procedures to approximately solve the partial differential equations that define the equilibrium of the continuum.

The first method, used by architects in the Middle Ages, was based only on equilibrium conditions. This applied restrictions to the path of the internal forces, caused by the weights imposed on the structure, consider that these paths had to be located within the central third of chosen critical sections where the bending moments had small magnitudes, allowing the problem to be statically determined.

The elastic-energy methods, unlike the force method considered above, consider, in addition to the balance equations, the equations of compatibility of deformations, using concepts of strain energy or elastic potential energy and the principle of virtual work, assuming that the behaviour of the material is elastic throughout all the process of deformation of the structure.

Due to recent technological developments, the methods of analyses to solve complex physical problems represented by ordinary or partial differential equations based on advanced numerical procedures have started to be more frequently used. Among the best-known procedures to solve these equations are the finite element, finite differences and boundary element methods. In this paper only the finite element method is utilized. This method allows, in the case of structural mechanics problems, the consideration of the complex constitutive models developed for the numerical simulation of the elastic and inelastic mechanical behaviour of the masonry considering linear and non-linear formulations of the geometry of the structure, and also the handling of complex shapes characteristic of historical structures such as vault roofs or arches.

The selection of the most adequate method of analysis to study the stability of masonry arches depends on load and boundary conditions in the structure. In many cases, as in the static analysis of an arch subjected to distributed vertical loads, an elastic analysis, or force equilibrium based method, is enough to define the thrust line; however, for situations with more complex load conditions or geometry, when it is difficult to identify which parts of the structure experience inelastic behaviour in the material, it is necessary to use more refined methods of analysis in order to obtain the distribution of stresses and the corresponding location of the thrust line. Furthermore, with this type of analysis, it is possible to consider the residual capacity of the structure to support the acting loads. Thus, it is very important to have a constitutive model for the material that correctly simulates its behaviour under different loading paths necessary to correctly calculate the response of the structure and to compare it with that observed in real structures.

4 NUMERICAL SIMULATION OF THE INELASTIC BEHAVIOUR OF MASONRY

In quasi-fragile heterogeneous materials, as is the case of masonry, non-linear behaviour is accentuated by strain localization due to the initiation and propagation of discontinuity zones even before the ultimate capacity of the material is reached, producing a reduction in the magnitude of the stresses with increasing strain. This phenomenon, known as strain softening, generally leads to an ill-posed boundary value problem due to the loss of ellipticity of the governing set of differential equations (De Borst *et al.*, 1994).

From a numerical point of view and within the context of the finite element method, the above situation is reflected as an extreme sensitivity of the solution procedure to the mesh configuration and characteristic element size used in the discretization of the structure, leading to a problem of the lack of convergence of the calculated response.

The description of the evolution of the stresses and strains in the inelastic range is frequently defined by mathematical functions that reflect the evolution of damage when the limit of elastic behaviour is exceeded. These functions are generally based on theoretical simplifications of the principles of the thermodynamics of irreversible processes.

Both the definition of the constitutive equations for the material and the solution of the non-linear equations that govern the deformation process in the continuum are important issues to be considered in the non-linear analysis with advanced numerical methods such as the finite element method. The convergence of the results to the correct global solution of the problem depends on the consistency of the procedures used for the solution of the non-linear systems of local and global equations.

The constitutive equations that describe the inelastic behaviour of the material can be defined from mathematical models based on the theory of plasticity, or on the concepts of smeared cracking or on continuum damage models to properly reflect the quasi-brittle characteristics of the masonry. In the case of models based on the theory of the plasticity, this paper uses, for the representation of the softening of the material, a tensile Rankine type orthotropic model of plasticity with a non-associated flow rule as proposed and used by Lourenço (1996). For the smeared cracking model the conventional approach is used where the material behaviour is assumed to be brittle. Finally, when the continuum damage approach is followed the Mazars model is used. The results obtained from the application of the above mentioned methods are compared to assess their real potential for the solution of this type of problem.

For the plasticity model, Lourenço (1996) proposed the following equation for the yield surface:

$$f = \frac{\left(\sigma_x - \bar{\sigma}_1(\kappa_t)\right) + \left(\sigma_y - \bar{\sigma}_2(\kappa_t)\right)}{2} + \sqrt{\left(\frac{\left(\sigma_x - \bar{\sigma}_1(\kappa_t)\right) + \left(\sigma_y - \bar{\sigma}_2(\kappa_t)\right)}{2}\right)^2 + \alpha\tau_{xy}^2} \tag{1}$$

where α is a factor used to calibrate the shear strength, and can be calculated with the following equation:

$$\alpha = \frac{f_{tx} f_{ty}}{\tau_u^2} \tag{2}$$

where f_{tx} and f_{ty} are the strengths in the x and y directions respectively and t_u is the strength in pure shear. σ_x, σ_y and τ_{xy} are components of the stress vector in a plane stress state. κ_t is an internal variable that defines the local characteristic of the inelastic behaviour of the material (Lubliner, 1990).

To describe the inelastic behaviour in the material it is important to consider its strain softening behaviour. The plastic potential function is described in a similar way as in eq. (1), but in this function the calibration factor is one.

Based on the stress-strain uniaxial diagram with softening, we assumed for plane state a linear of the stresses for the softening part in both orthogonal material directions as follows:

$$\bar{\sigma}_1(\kappa_t) = f_{tx}\left(1 - \frac{hf_{tx}}{2G_{fx}}\kappa_t\right), \quad \bar{\sigma}_2(\kappa_t) = f_{ty}\left(1 - \frac{hf_{ty}}{2G_{fy}}\kappa_t\right) \tag{3}$$

where h is the equivalent element length needed to have the mesh objectivity (Bazant and Planas, 1998) and G_{fx} and G_{fy} are the fracture energies in the x and y directions, respectively.

A powerful and widely used approach in the finite element analysis of quasi-brittle materials is the concept of smeared cracking, originally introduced by Rashid in 1968. This concept considers that the stress in the finite element must decrease when the tensile strength is reached. Initial models assumed a brittle behaviour, however, results proved that this assumption was not strictly valid as the tensile stress is reduced gradually after its maximum value is reached. Although this approach has been implemented in large finite element programs, it was only recently found that the finite element size, for an objective solution, has an important influence on the convergence of the solution (Bazant and Oh, 1983; Rots, 1988 and Bazant and Planas 1998). In spite of this, for the analysis of historical buildings it is still common practice to consider the masonry as a brittle material, i.e., stresses drop suddenly to zero without internal energy dissipation when the tensile strength is exceeded. This paper takes into account this last behaviour in the illustrative example.

A more appropriate alternative to simulate the inelastic behaviour of quasi- brittle materials, very often implemented in programs of non-linear analysis which simulate the response and failure modes of masonry, is to use models based on continuum damage theories. These theories as is also the case of the theory of plasticity, are based on the thermodynamics of the irreversible processes and on the theory of internal state variables. Physically, the degradation of the mechanical properties of the material is attributed to the growth of micro-defects. Simo and Wu (1986) described two alternative frameworks, strain- and stress- based, to develop continuum damage models. In the strain-space based formulation, to model the isotropic damage process, the hypothesis of strain equivalence allows the definition of the following equation for the effective stresses:

$$\sigma_{ij}^{damage} = (1 - d)\sigma_{ij}^{effective} \tag{4}$$

where d is a scalar damage parameter.

On the other hand, the stress space formulation involves the notion of equivalent strain and characterizes the damage by the notion of effective strain together with the stress equivalence. In this case the effective strain is expressed as:

$$\varepsilon_{ij}^{effective} = (1 - d)\varepsilon_{ij}^{damage} \tag{5}$$

The essential characteristic of these strain- and stress –based damage models is the remarkable simplicity of their numerical implementation within the context of finite element methods. As an example, Mazars (1981 and 1986) developed a series of damage models to study the behaviour of concrete structures damaged under tension and under compression. The proposed damage parameter is:

$$d = d_t\alpha^{\beta} + d_c(1 - \alpha)^{\beta} \tag{6}$$

The parameter α is given by the ratio $\varepsilon^{eq}/\varepsilon$, where ε^{eq} represents the positive principal strains, ε represents the total strains (both negative and positive) and $\beta = 1$. The values of d_t and d_c can be obtained from:

$$d_t = 1 - (1 - a_t)\frac{\varepsilon^{o}}{\varepsilon^{eq}} - a_t e^{-b_t(\varepsilon^{eq} - \varepsilon^{o})} \tag{7}$$

and

$$d_c = 1 - (1 - a_c)\frac{\varepsilon^{o}}{\varepsilon^{eq}} - a_c e^{-b_t(\varepsilon^{eq} - \varepsilon^{o})} \tag{8}$$

where ε^{o} is the strain threshold for damage. Typical values for other material parameters for concrete used by Mazars are shown in Table 1.

Table 1 : Parameter values for damage model proposed by Mazars

	Lower limit -	Upper limit -
ε^{o}	0.0001	0.0003
a_t	1.0	1.5
b_t	500.0	2000.0
a_c	0.7	1.2
b_c	e^4	$5e^4$

5 APPLICATION EXAMPLE

5.1 Characteristics of the model

In order to compare the different models for inelastic material behaviour one of the arches of the Metropolitan Cathedral of Mexico City was chosen. This arch is shown shaded in Figure 1.

The numerical models used consider the material as isotropic. The value of Young's modulus is 2.5E6 KN/ m^2 and the Poisson ratio is 0.18. In the plasticity model the fracture energy is 1.2E6 KN/ m^2. The masonry tensile strength is 280 KN/m^2 and the strain threshold for damage is 0.0001. The arch is considered subjected to a uniform vertical load of 19.6 KN/m representing the self-weight of the structure, and outward horizontal displacements of the supports of 80.0 mm were imposed to represent the horizontal movements of the support columns of the arch.

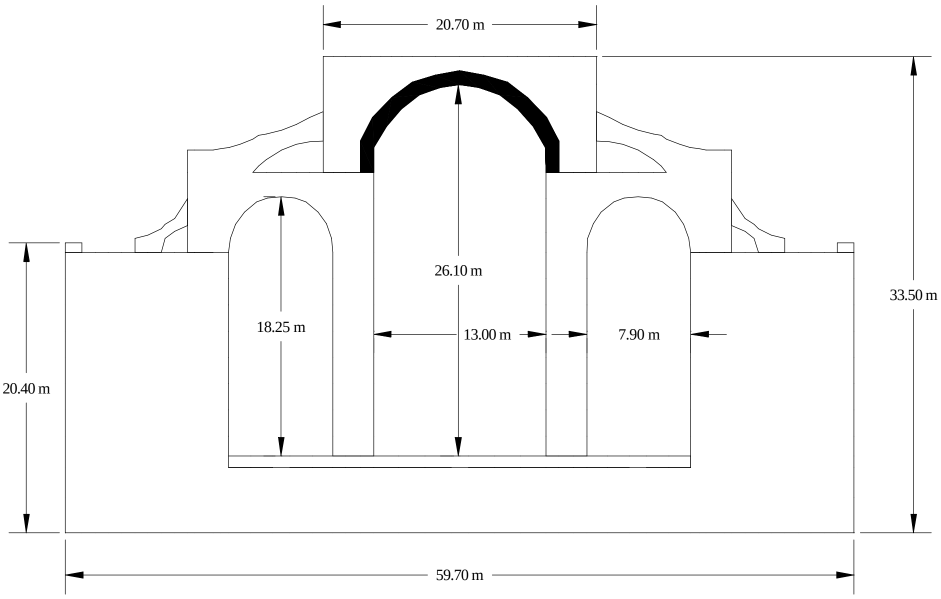

Figure 1 : Typical transversal section of the Metropolitan Cathedral of Mexico City.

5.2 Results of the elastic analysis

The linear elastic analysis of the arch was done using the flexibility method. The first model was subjected to the uniform vertical loading that simulates the self-weight of the arch. The location of the thrust line for this case is depicted in Figure 2. Roeder (1998) reported that results for the same load conditions, from the practical point of view, do not significantly change when compared with those obtained with the non-linear elastic model.

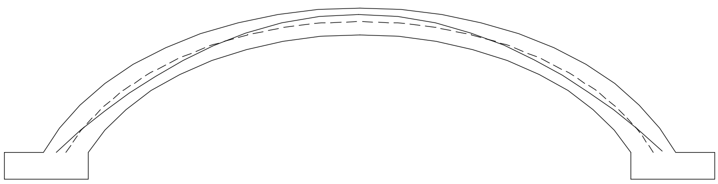

Figure 2 : Thrust line for vertical uniform load.

5.3 Results of the inelastic analysis

In this case, the magnitude of the compression force and its position on sections normal to the axis of the arch are evaluated numerically integrating the stress distributions obtained from the analysis with nine node lagrangian finite element meshes. The results from smeared cracking models were obtained with the program DIANA (TNO, 1996). The analysis with Rankine plasticity models was carried out with the program ANSA-IINGEN (Roeder, 2000) using a non-symmetrical sparse linear solvers available in the Harwell Subroutine Libraries (AEA Technology, 2001) as this model leads to non-symmetrical finite element jacobian matrices. The analysis with the damage model of Mazars was carried out using the TOCHNOG program written at the University of Twente in Holland (Roddeman, 2000).

Figure 3 shows the thrust lines for an arch with brittle behaviour, obtained using smeared cracking models.

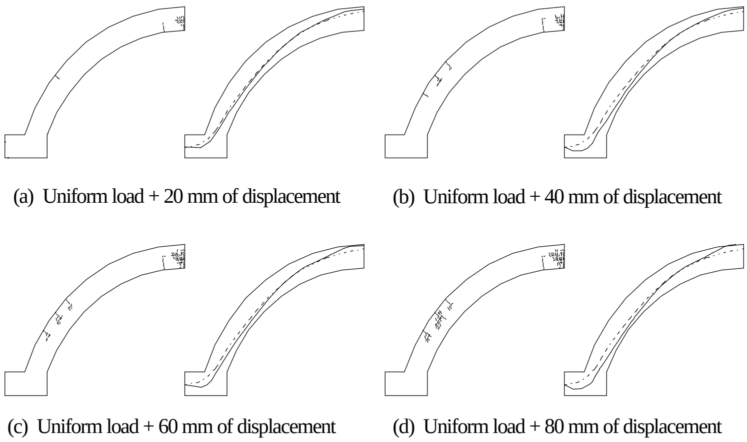

Figure 3 : Thrust lines for uniform load and outward support displacements using a brittle material.

Thrust lines obtained from the non-linear analyses of the arch corresponding to the three material behaviour models are illustrated in Figure 4.

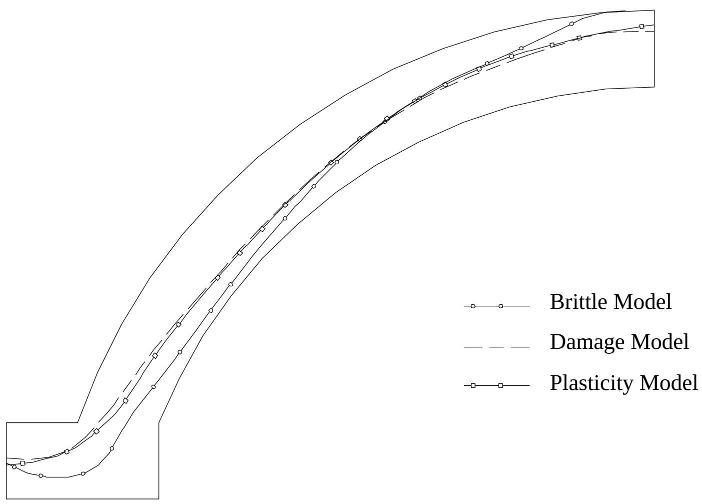

Figure 4 : Thrust lines for uniform load and -displacements obtained from non-linear analyses.

Figures 5 and 6 show the material second order work, $d\sigma_{ij}d\varepsilon_{ij}$ (Bazant and Cedolin, 1991) , for the Rankine plasticity model to see the material instabilities.

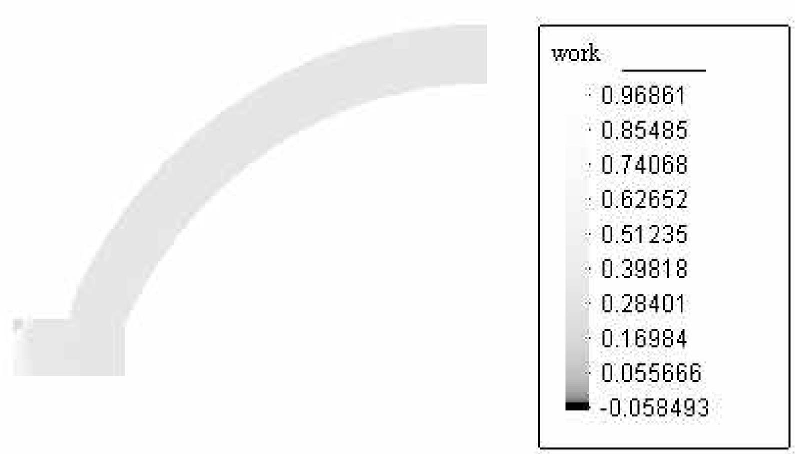

Figure 5 : Material second order work for uniform load and outward support displacements.

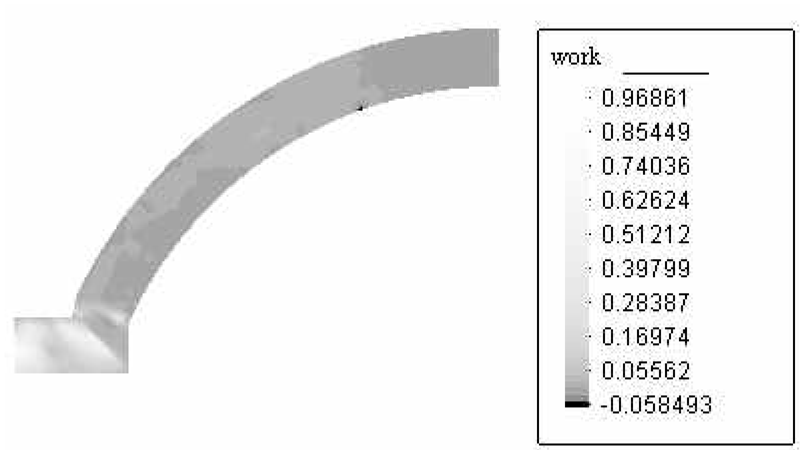

Figure 6 : Material second order work for outward support displacements.

6 CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE WORK

In this paper the results reported with the different models considered in this paper were compared.

In the brittle model the location of the thrust line is almost on the limits of the arch geometry, whereas in the inelastic models, under the same conditions, is still within the arch geometry. This result implies that for brittle models the instability conditions are reached more rapidly than for the inelastic models. For the inelastic models the difference between the results from plasticity and continuum damage models is negligible. This result indicate that, in general, if the quasi-fragile nature of the material is considered, both inelastic models correctly predict the stability of the arch under complex loads conditions. Despite these results, the inelastic models are computationally more expensive to use than the brittle models.

The rate of loss of the accumulated second order work in the inelastic models subjected to uniform load and outward horizontal displacements is smaller than that corresponding to the model subjected only to outward horizontal displacements. This is due to differences in the extent of softening zones in the masonry that define the correct equilibrium state of the structure, and also show that the assumption that the behaviour of the masonry is only brittle is not strictly correct but that some additional geometric considerations must be taken into account.

It is the authors' opinion that future work should be directed to improve the models- of non-linear analyses and also to develop simpler structural models and methods of analysis aimed to give, for practical applications, the good results at a reduced cost. The inelastic models are still the best, however, for their correct application requires more control during the analysis phase drastically incrementing the computational cost of its use. The use of brittle models is not recommended as the results are as good as those corresponding to the inelastic models.

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