Constitutive behaviour of masonry

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ABSTRACT: Load bearing masonry structures often exhibit some degree of confining, which add global "ductility" to the structure. For this reason elastic-plastic analyses of masonry structures may provide fair approximations of the stress and strain field. Moreover, the plasticity theory is general enough to be applied to materials, like masonry, which present strong material anisotropy. The homogenisation techniques, largely employed in the case of composite materials and recently applied to masonry, provide in fact as a result a macroscopic equivalent medium with anisotropic characteristics, due to the preferential orientation of the joints. According to this considerations, this paper presents an approach to the development of a constitutive model for masonry, developed in the framework of rate independent softening plasticity, involving an anisotropic yield criterion based on the classical Mohr-Coulomb criterion, taking into account the material orthotropy by means of a friction tensor.

1 INTRODUCTION

The knowledge about masonry, the most used building material throughout the world, is still incomplete. For a long time codes of practice have proposed geometrical rules that have been the only reference to designers.

Some of the research efforts made in recent years to achieve a better understanding of masonry constitutive behaviour have been developed in the framework of plasticity (Pietruszczak and Niu, 1992). Constitutive models developed for concrete (Feenstra et al., 1996) have been successfully applied to masonry structures. Unfortunately the postulates by Drucker (1951 and 1988) and Hill (1958) providing the framework for establishing the material stability have in recent times come under very strong challenge in relation to their validity for geomaterials. With this word are often termed the materials, such as soils, rocks, concrete and masonry, which exhibit frictional behaviour. The use of non-associated flow rules has been proposed as a way of accomodating such class of materials that, while experiencing dilation during shearing, nonetheless remain stable (Salençon, 1975, Famiyesin, 2001).

In computational plasticity the normality condition is assumed due to the numerical convenience and to the more developed theory (Simo and Taylor, 1985). Those who adopt the standard model argue about the inadmissibility of non-associated flow from thermodynamic point of view (Runesson and Sture, 1989).

Recently the theoretical and numerical problems about the variational formulation of non-associated law have been developed (Polizzotto, 1997, Collins and Houlsby, 1997, Famiyesin, 2001), although the complexity of the problem welcomes more studies in the future.

Load bearing masonry structures often exhibit some degree of confining, which add global "ductility" to the structure. For this reason elastic-plastic analyses of masonry structures may provide fair approximations of the stress and strain field (Genna et al., 1998). Moreover, the plasticity theory is general enough to be applied to materials, like masonry, which present strong material anisotropy (Pietruszczak et al., 1988, Lourenço et al., 1995 and 1997). The homogenization techniques (Nemat-Nasser, 1993, Aboudi, 1990), largely employed in the case of composite materials and recently applied to masonry, provide in fact as a result a macroscopic equivalent medium with anisotropic characteristics, due to the preferential orientation of the joints (Anthoine, 1992, De Buhan and De Felice, 1997). According to this considerations, this paper presents an approach to the development of a constitutive model for masonry, involving an anisotropic yield criterion based on the classical Mohr-Coulomb criterion, taking into account the material orthotropy by means of a friction tensor (Frunzio et al., 2000).

2 THE CONSTITUTIVE MODEL

Reference is made to an anisotropic inelastic body with softening constitutive law, defined over a domain Ω on which loads $\mathbf{t}_0$ and displacement $\mathbf{u}_0$ are assigned respectively on the parts $\partial\,\Omega_u$ and $\partial\,\Omega_t$ of the boundary $\partial\,\Omega$, while body forces $\mathbf{b}_0$ are assigned in Ω as in figure 1. The governing relations are developed with the assumptions that strain and displacements are small.
The static and kinematic fields are referred to a Cartesian co-ordinate system:

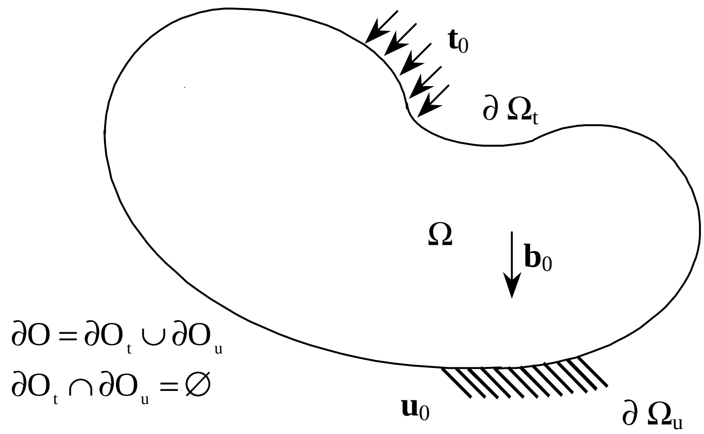

Fig. 1: Inelastic body

With the above assumptions,

the compatibility conditions for the body Ω give:

$$\begin{aligned} &\mathbf{e} = \mathrm{Sym}\nabla\mathbf{u} \\ &\mathbf{u} = \mathbf{u}_0 \quad \text{on} \quad \partial\Omega_u \end{aligned} \tag{1}$$

where $\mathbf{e}$ represents the small strain tensor. The equilibrium conditions for the body Ω give:

$$\begin{aligned} &\mathrm{div}\ \ \mathbf{s} + \mathbf{b}_0 = 0 \\ &\mathbf{s}\ \ \mathbf{n} = \mathbf{t}_0 \qquad\qquad \text{on} \quad \partial\,\Omega_t \end{aligned} \tag{2}$$

where $\mathbf{s}$ represents the Cauchy stress tensor and $\mathbf{n}$ is the outward normal.

the constitutive problem for the body Ω is governed by the following relations:

$$
\begin{aligned}
&\mathbf{s} = \frac{\partial \omega(\mathbf{e})}{\partial \mathbf{e}} && \text{in } \Omega \\
&\mathbf{u} = \mathbf{u}^{e} + \mathbf{u}^{i} \\
&\mathbf{t} = \frac{\partial ?\,(\mathbf{u}^{e}, \mathbf{a})}{\partial \mathbf{u}^{e}} && \mathbf{s} = \frac{\partial ?\,(\mathbf{u}^{e}, \mathbf{a})}{\partial \mathbf{a}} \\
&\dot{\mathbf{u}}^{i} = \frac{\partial \mathbf{g}\,(\mathbf{s}, \mathbf{s})}{\partial \mathbf{s}}\,\dot{?} && \dot{a} = \frac{\partial \mathbf{g}\,(\mathbf{s}, \mathbf{s})}{\partial \mathbf{s}}\,\dot{?} \\
&\dot{?} \geq \mathbf{0} && \mathbf{f}(\mathbf{s}, \mathbf{s}) \leq \mathbf{0} \qquad \mathbf{f} \cdot \dot{?} = 0
\end{aligned}
\tag{3}
$$

where ω is the strain energy density function corresponding to an elastic behaviour for the body Ω. If a linear elastic behaviour is assumed, ω is a quadratic function of $\mathbf{e}$; the displacement $\mathbf{u}$ is considered as the sum of an elastic part $\mathbf{u}^e$ and an inelastic part $\mathbf{u}^i$. The behaviour of the body is defined in two conjugate spaces of generalized variables: the set of internal kinematic variables $\mathbf{a}$ and the conjugate set of statical internal variables $\mathbf{s}$ define the inelastic behaviour, while the elastic behaviour is defined by the two conjugate kinematic variable $\mathbf{u}^e$ and static variable $\mathbf{s}$. The plastic multipliers are expressed by $\mathbf{?}$.

The tractions $\mathbf{t}$ and the static internal variables $\mathbf{a}$ are given as derivatives of a free energy function density $?\,(\mathbf{u}^e, \mathbf{a})$, non convex to reproduce softening. The yield function $\mathbf{f}(\mathbf{s}, \mathbf{s})$ is a different function from the plastic potential $\mathbf{g}(\mathbf{s}, \mathbf{s})$. This corresponds to the assumption of nonassociated plasticity, this corresponds to the definition of masonry as a generalized non standard material. The problem to be solved is to find the equilibrium configurations of the body Ω under the assigned loads.

2.1. The yield criterion

As yield criterion can be considered that one proposed by the authors for anisotropic frictional materials (Frunzio et al., 2000), of the form:

$$
\mathbf{f} = \left\| \left(\mathbf{I} - \mathbf{m} \otimes \mathbf{m} \right) \mathbf{Tm} \right\| - (c - \mathbf{Tm} \otimes \mathbf{m}) \left\| \mathbf{Mm} \right\| \leq 0 \tag{4}
$$

where the friction coefficient $\mathrm{tg}\,\varphi(\mathbf{m})$ on a plane with unit normal $\mathbf{m}$, varies according to the following tensor law:

$$
\mathrm{tg}\varphi(\mathbf{m}) = \left\| \mathbf{Mm} \right\| = \left\| \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} \right\|
$$

the symbol $\|\bullet\|$ represents the Euclidean norm and:

$$
\mathbf{M} = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix}
$$

is the friction tensor in which $\mu_i = \mathrm{tg}\varphi_i$ is the friction coefficient and φ_i is the friction angle. The cohesion $c\ \mathrm{tg}\varphi(\mathbf{m})$ changes according to the variation of friction coefficient, while c is a material parameter. The principal friction planes in a masonry panel coincide with the midplane of the panel, and the two orthogonal planes of bed and head joints.
The stress tensor $\mathbf{T}$ is given by:

$$\mathbf{T} = \mathbf{Q}^{\mathbf{T}}\ \mathrm{s}\ \mathbf{Q}$$

where the tensor **Q** represents the change of reference from the friction reference (related to the texture of the masonry panel) and the cartesian coordinate system to which the static and the kinematic field are referred.
The set of admissible states according to (4) is convex (Gesualdo and Monaco, 1998). It must be noted that the problem of convexity of the yield domain is a fundamental question. In the cases of non standard materials, where the convexity cannot be derived from the postulate by Drucker, a non convex domain requires appropriate loading paths for the solution of the problem while the convexity assures computational advantages (Lemaitre and Chaboche, 1990, Maugin, 1992).

2.2. The plastic potential

As plastic potential can be considered a function formally similar to that proposed for the yield criterion, where the friction tensor is replaced by a "dilatancy tensor" (Salencon, 1975), of the form:

$$\mathbf{g} = \left\| (\mathbf{I} - \mathbf{m} \otimes \mathbf{m}) \mathbf{Tm} \right\| - (c - \mathbf{Tm} \otimes \mathbf{m}) \left\| \mathbf{Nm} \right\| \leq 0 \tag{5}$$

where the dilatancy tensor is given by:

$$\mathbf{N} = \begin{bmatrix} \nu_1 & 0 & 0 \\ 0 & \nu_2 & 0 \\ 0 & 0 & \nu_3 \end{bmatrix}$$

and $\nu_i \leq \mu_i$.

2.3. Weak formulation

A numerical solution of the problem is possible, so that a weak formulation of the problem, that allows the spatial discretization through finite elements, is proposed. Assuming that the compatibility and the constitutive laws are locally fulfilled on the body **Ω**, a weak formulation involves the relations (6) and (7). The equilibrium conditions can be expressed by the classical formulation of the virtual work:

$$\int_{\Omega} \mathbf{s}(\mathbf{e}(\mathbf{u})) \cdot \delta\mathbf{e}(\mathbf{u})\ d\Omega - \int_{\Omega} \mathbf{b}_0 \cdot \delta\mathbf{u}\ d\Omega - \int_{\partial\Omega_t} \mathbf{t}_0 \cdot \delta\mathbf{u}\ dS = 0$$

$$\mathbf{u} \in U\ ,\quad \forall\ \delta\mathbf{u} \in U_0 \tag{6}$$

where

$$U = \{\mathbf{u} : \mathbf{u} \text{ regular in } \Omega,\ \mathbf{u} = \mathbf{u}_0 \text{ on } \partial\Omega_u\}$$

$$U_0 = \{\delta\mathbf{u} : \delta\mathbf{u} \text{ regular in } \Omega,\ \delta\mathbf{u} = \mathbf{0} \text{ on } \partial\Omega_u\}\ .$$

While the constitutive equations are given by:

$$\int_{\Omega} (\mathbf{s} - \frac{\partial \psi}{\partial \mathbf{u}^{e}}) \cdot \delta \mathbf{u}^{e} \; d\Omega = 0 \quad , \quad \forall \; \delta \mathbf{u}^{e} \qquad (a)$$

$$\int_{\Gamma} (\mathbf{s} - \frac{\partial \psi}{\partial \mathbf{a}}) \cdot \delta \mathbf{a} \; d\Omega = 0 \quad , \quad \forall \; \delta \mathbf{a} \qquad (b) \qquad (7)$$

$$\int_{\Omega} (\dot{\mathbf{u}}^{i} \cdot \delta \mathbf{s} + \dot{\mathbf{a}} \cdot \delta \mathbf{s}) \; d\Omega \leq 0 \quad , \quad \forall \; \delta \mathbf{s} , \delta \mathbf{s} \in T \qquad (c)$$

where

$$T = \{ \delta \mathbf{s} = \mathbf{s}^{*} - \mathbf{s} , \delta \mathbf{s} = \mathbf{s}^{*} - \mathbf{s} : \; \mathbf{f}(\mathbf{s}^{*}, \mathbf{s}^{*}) \leq 0 \}$$

The above relations can be obtained solving an extremum problem for the functional:

$$L(\mathbf{s}, \mathbf{s}, \dot{\mathbf{?}}) = -\int_{\Omega} (\mathbf{s} \cdot \dot{\mathbf{u}}^{i} + \mathbf{s} \cdot \dot{\mathbf{a}}) \; d\Gamma + \int_{\Omega} \dot{\mathbf{?}} \cdot \; \mathbf{g}(\mathbf{s}, \mathbf{s}) \; d\Omega \qquad (8)$$

The sub-stationariness condition of the functional $L(\mathbf{s}, \mathbf{s}, \dot{\mathbf{?}})$ furnishes in fact a point $(\mathbf{s}, \mathbf{s}, \dot{\mathbf{?}})$ that satisfies the integral relations in (9).
The structure of the functional *L* has recently studied with specific reference to generalized non standard materials (Polizzotto, 1997).
The formulation can be the basis for space discretization by means of finite elements, replacing the field variables $\mathbf{s}$, $\mathbf{u}$, $\mathbf{s}$ and $\mathbf{a}$, conjugate in pairs, with the interpolations of the corresponding nodal quantities (Bolzon and Corigliano, 1997).

$$\int_{\Omega} (\dot{\mathbf{u}}^{i} - \frac{\partial \mathbf{g}}{\partial \mathbf{s}} \; \dot{\mathbf{?}}) \cdot \delta \mathbf{s} \; d\Omega = 0 \quad , \quad \forall \; \delta \mathbf{s} \qquad (a)$$

$$\int_{\Omega} (\dot{\mathbf{a}} - \frac{\partial \mathbf{g}}{\partial \mathbf{s}} \; \dot{\mathbf{?}}) \cdot \delta \mathbf{s} \; d\Omega = 0 \quad , \quad \forall \; \delta \mathbf{s} \qquad (b) \qquad (9)$$

$$\int_{\Omega} \mathbf{f}(\mathbf{s}, \mathbf{s}) \; \delta \dot{\mathbf{?}} \; d\Omega \leq 0 \quad , \quad \forall \; \delta \dot{\mathbf{?}} \in M \qquad (c)$$

where

$$M = \{ \delta \dot{\mathbf{?}} = \dot{\mathbf{?}}^{*} - \dot{\mathbf{?}} : \; \dot{\mathbf{?}} \neq \mathbf{0} \text{ on } \Omega \}.$$

3 CONCLUSIONS

This paper has presented an approach to the development of a constitutive model for masonry, involving an anisotropic yield criterion based on the classical Mohr-Coulomb criterion, taking into account the material orthotropy by means of a friction tensor. The framework of plasticity with internal variables provides a good description of the softening problem, which can represent the behaviour of masonry. A numerical solution of the problem, by means of spatial discretization through finite elements, is possible.

ACKNOWLEDGMENTS

The support of the Italian Ministry of University and Scientific Research (M.U.R.S.T.) is gratefully acknowledged.

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