Limit state analysis of hemispherical domes with finite friction

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ABSTRACT: While it is recognised that often masonry vaulted structures fails by way of sliding, because this condition involves violation of the normality rule for plastic analysis, most of the existing procedures assume that sliding is prevented by sufficiently high friction at the interface between blocks or voussoirs and that failure is achieved by relative rotation of blocks. The inclusion of the sliding mechanism poses the problem of the uniqueness of the solution found applying standard limit-state analysis procedures. The paper first provides a simple proof of the uniqueness based on the condition of symmetry and equilibrium for domes with general load conditions. The proof of the unique solution allows a straightforward treatment of this class of problems as one of standard limit-state analysis.

On these bases a computer procedure, using a lower bound approach, has been developed to define the condition of stability of such structures. Particularly for domes this procedure leads to determine admissible membrane surfaces within the thickness.

1 INTRODUCTION

Vaulted structures are defined as form active structures, i.e. structures that derive their capacity from the distribution of material in space. Hence during the past centuries architects and engineers have refined the distribution of masonry material to obtain increasingly robust structures. Of course this has not been a smooth process and numerous have been the failures within it. In the case of domes, which is the subject of this paper, this eventually resulted in elliptical generating arches, with rise greater than the radius of the dome, and in thickness tapering from the spring of the dome to the apex. Double shells and ribs were also introduced to optimise the weight to span ratios.

From a geometrical point of view the process can be seen as the definition of a smooth curve (the generating arc) and the allocation of material around that arc, defining the thickness of the generating arch. The best allocation of material is constrained by its mechanical properties, and in the case of the masonry this means to exploit its high compressive strength, and use the friction resistance generated by the compressive action to withstand tensile stresses.

This paper presents a limit state analysis approach to define the minimum constant thickness required for a hemispherical masonry dome to withstand a weight-like load distribution. Previous work considered either the membrane hypothesis, assuming that the tensile hoop stress generated by the shape are borne by the material tensile strength or, the other extreme hypothesis, that the material is not able to accommodate hoop stresses, leading to the analysis of the stability of a dome with meridian cracks. In the first case the solution is independent of the thickness, as no limitation is posed to either compressive or tensile stresses. In the second case the problem reduces to the one of an arch of increasing width from the crown to the spring, and the value of minimum thickness to span ratio as a safe solution was first provided by Heyman (1977) under the assumptions of infinite compressive strength and friction resistance and zero tensile strength.

Oppenheim et al. (1989), have used the assumption of zero hoop stresses to define a closed form solution of the fundamental differential equilibrium equations for the case of hemispherical domes without oculus, under axisymmetrical loading conditions. This reduces the problem to a one dimensional one, and provides lower bound solutions, but does not optimise the material distribution.

Recently some new approaches have been developed. Among these it is worth mentioning O'Dwyer (1999) that proposes a solution based on limit state analysis and linear programming. The principal stresses in a vaulted structure (groin vaults or domes) are modelled as a discrete network of forces. However the solution proposed is based on a geometric limitation for the position of the resultant lines of thrust and on an initial assumption of a value for the horizontal component of the resultant of stresses. The assumption for the material is of no tensile strength but friction high enough to prevent sliding. Therefore mechanisms associated with sliding failure are excluded a-priori, but at the same time the fictitious tensile strength derived from friction is not taken into account. As a result, the solutions obtained, while might be safe, are neither unique or the maximum.

Villaggio (2000) has instead proposed solutions for domes loaded with a concentrated force perpendicular to the axis of revolution. The aim is to identify the best shape for the generating arch, assuming unit height and unit volume, and hence providing a law for the best distribution of material. However the dome is modelled as a membrane and no limitation is posed on the tensile hoop stresses. This is therefore of limited use for the analysis of masonry domes.

Of particular interest with respect to the work presented here are general approaches for the limit state analysis of block masonry structures, in which the possibility of finite friction is contemplated. In this case the material behaviour is non standard in terms of plasticity theory. The satisfaction of the normality rule will imply the presence of dilatancy, which does not occur in reality, and non-compliance with the normality rule means that the fundamental theorems of plasticity will not in general provide a unique solution. Livesley (1978, 1992), by adopting a lower-bound approach, was the first to developed a formal linear programming procedure to define the maximum load factor of two and three-dimensional structures. Particularly for domes further developments are proposed by D'Ayala (1993, 1994). Lo Bianco and Mazzarella (1983) have proposed a procedure based on mathematical programming that satisfies uniqueness and does not imply dilatancy. Recently Baggio and Trovalusci (1998a, 1998b) have proposed a non-standard limit analysis approach for 2D and 3D block masonry structures based on determining the minimum of a class of load factor satisfying the kinematical and static condition simultaneously. The associated programme results however rather onerous in terms of time and memory requirements, as it depends on non-linear and non-convex optimisation procedures.

Casapulla (1999) together with Jossa (1998, 2001) have discussed classes of problems for which unique or safe solutions within limit state analysis can be found. A generalisation to 3D systems, by studying the behaviour of the block interface, has been presented by Casapulla and D'Ayala (2001), with application to barrel vaults and, in this paper, it is extended to the analysis of axisymmetric masonry domes. The procedure developed is based on membrane theory used within limit state analysis. While the conventional membrane theory assumes the membrane stress surface to coincide with the geometric surface, in the present work the admissible membrane stress surface will be required only to lie within the dome thickness and satisfy frictional constraints. Therefore, there can be more than one such membrane surface, and these in general will differ from the middle surface of the dome.

After proving the uniqueness of the solution at a limit state, due to the symmetry, the optimum surface providing the minimum thickness of the dome is herewith defined by using mathematical programming.

2 AXISYMMETRIC DRY BLOCK MASONRY DOMES

2.1 Uniqueness of the solution

Masonry domes can be modelled as three-dimensional discrete systems of rigid blocks in dry contact, under the assumptions of infinite compressive strength for the blocks, no tension transmitted across the joints and shear strength at blocks interfaces determined by cohesionless Coulomb friction. The blocks do not necessarily represent the actual fabric of the masonry constituting the dome, as this can be made out of brickwork, but each block can be considered as a macroelement of homogenous material.

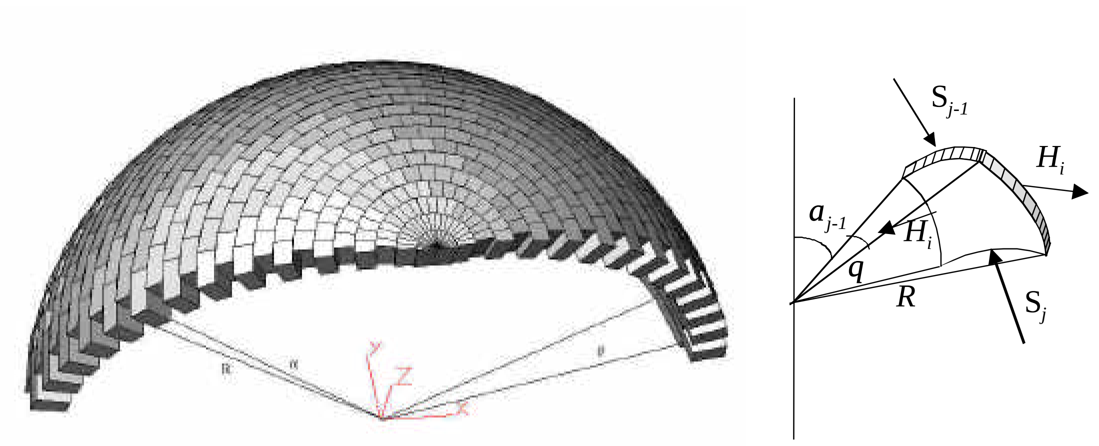
The analysis is therefore conducted on interfaces, whose state of stress is governed by frictional (plastic) constraints. Safety under the stated material assumptions is assured if a membrane surface can be found which everywhere lies within the thickness of the dome and satisfies frictional constraints. This resistant surface is characterised by meridian and hoop stresses but its profile does not in general coincide with the mean hemispherical surface. This means that the curvature of its generating meridian is not known a–priori and is generally not constant.

Consider the axisymmetric dome, of which only half is represented in Fig. 1, subject to its own weight and formed from rigid blocks of wedge shape, with staggered meridian joints. The interface between two adjacent blocks in the meridian direction lies on a radial plane and the contact forces may have components both normal and tangential to the interface, the latter being resisted by friction. Moreover due to the shape of the dome hoop stresses will develop among adjacent blocks along a parallel. From general shell theory it is already known that under this loading condition the hoop stresses will be of compression in the upper portion of the dome and tensile in the lower portion. The problem is from a static point of view a standard membrane problem, and hence it is isostatic and the two components of stresses along the meridian and parallel directions can be defined as follows, using general shell theory:

$$S_j = \frac{W}{2px_j \sin g_j} \quad \text{and}$$

$$H_i = \frac{W_j * \cos((g_{j-1} + g_j)/2) - (S_{j-1}x^m_{j-1} + S_j x^m_j)\sin((g_j - g_{j-1})/2)}{\sin((g_{j-1} + g_j)/2)} \tag{1}$$

where W is the total weight of the portion of dome identified by an angle a_j from the vertical axis Z, S_j is the meridian stress for unit length of parallel and H_i is the resultant hoop stress for the length of meridian of the membrane comprised by two consecutive level at which S_j is calculated.



(a) (b)

Figure 1: (a) The masonry half dome. (b) Stress resultants for a discrete element of surface.

W_j is the weight of the element of dome considered (Fig. 1b), $x^m{}_j$ and g_j are the horizontal distance of the membrane from the z axis and the angle that the tangent to the membrane forms with the horizontal axis at each interface, respectively. The values of γ are therefore the geometric unknowns that define the equilibrium surface within the geometrical dome.

In order to prove the uniqueness of the solution, the two resultant of stresses in (1) for a given wedge of angle q of the dome can be considered, obtaining a system of forces for which the load condition and the resultants of stress are all contained in a vertical plane as in Fig.2. Said $\underline{S_j}$, $\underline{H_j}$ and $\underline{W_j}$ the resultant of the meridian stresses at interface j of the lune defined by angle a_j in the meridian plane and of angular width q ; the hoop stress resultant at this level; and the self-weight of this portion of the dome, respectively; by drawing at this interface the projection of the cohesionless Coulomb's cone (shaded in grey in Fig.2), it is evident that the range of admissible values for the maximum stress resultant $\underline{S_r}$ has a lower bound depending on the weight and the angle of friction j:

$$\underline{S_r} \geq \frac{\underline{W_j}}{sin(a_j + j)} \tag{2}$$

where the equality defines point A on the Coulomb's cone projection (Fig. 2), implying the incipient outward sliding of the lower portion of the lune with respect to the upper one. The shape of the structure and the assumption of rigid bodies prevent the opposite direction of sliding.

The limiting compressive and shear force resultants are statically defined, according to the membrane theory, and are obtained by summing the two membrane stress resultants $\underline{S_j}$ and $\underline{H_j}$ in the directions normal and tangential to the meridian, as follows:

$$\begin{aligned} \underline{N_j} &= \underline{S_j}\cos(a_j - g_j) - \underline{H_j}\cos a_j = \frac{\underline{W_j}}{\sin(a_j + j)}\cos j \\ \underline{T_j} &= -\underline{S_j}\sin(a_j - g_j) + \underline{H_j}\sin a_j = \frac{\underline{W_j}}{\sin(a_j + j)}\sin j \end{aligned} \tag{3}$$

The limiting values of T and N given by relations (3) are independent of each other and only depend on the self-weight and on the given value of f. This means that, in case of symmetric loading, there is a unique limiting value of the shear force, and the local equilibrium problem is at a limit state statically determined.

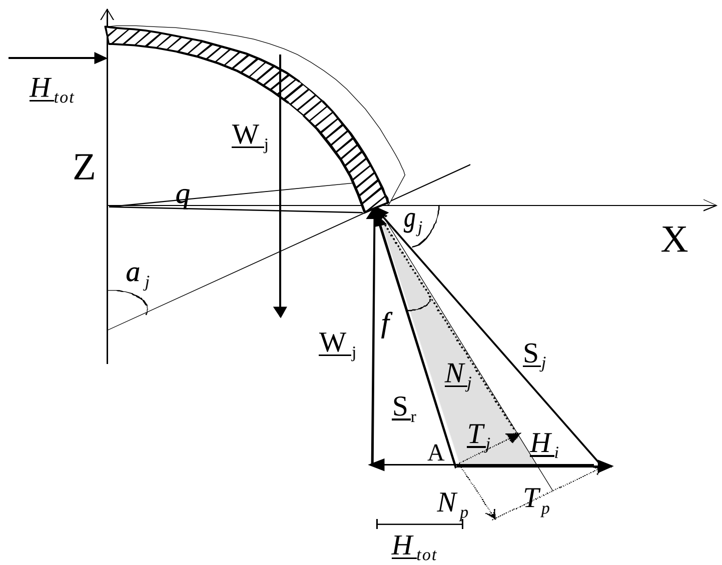


Figure 2 : The limiting membrane stress resultants in the meridian plane of a lune of dome.

This derives from the fact that, even for non standard materials, if normal forces are given at a limit state (conditions (3)), then they can be ignored in defining the yield surface, and, consequently, the Coulomb's cone reduces to a circle in the plane of shear forces (Goyal et al. 1991). The size of the circle obviously depends on the magnitude of the normal force, but the imposition of the normality rule now does not implies dilatancy, and hence the solution, being equilibrated, at the yield surface, and not violating the kinematical constraints, is indeed the correct solution and is unique. Therefore the material constraints become now standard and the analysis falls within the framework of the classical plasticity theory.

2.2 Static approach with limit-state analysis for a block masonry dome

The uniqueness proof allows to use a standard static approach within limit state analysis to calculate the minimum thickness of a dry block masonry dome of hemispherical shape and given mean radius, under its self weight. The procedure is developed for electronic spreadsheet and a multi-purpose mathematical programming solver is used to solve the problem of optimum.

Given the symmetry of the problem, only half of the dome is considered in the numerical procedure, and this is modelled by assuming that its mass is distributed along its middle hemispherical surface of radius *R* (its projection on the plane XZ is represented by a continuous curved line in Fig. 3a). In the same figure, forces W_i’s define the total weights of each course of the half dome (with i = 1 to n from the apex down), and the dashed curve line represents the locus of the centres of gravity of each half dome course. Hence, if x_i^R and z_i^R identify the centre of gravity of the unit weight W_i^R, (for unit length of the parallel arc), then the centre of gravity for the total weight of the corresponding half domical ring has same ordinate z_i^R and abscissa x_i (centre of gravity of the half circumference of radius x_i^R):

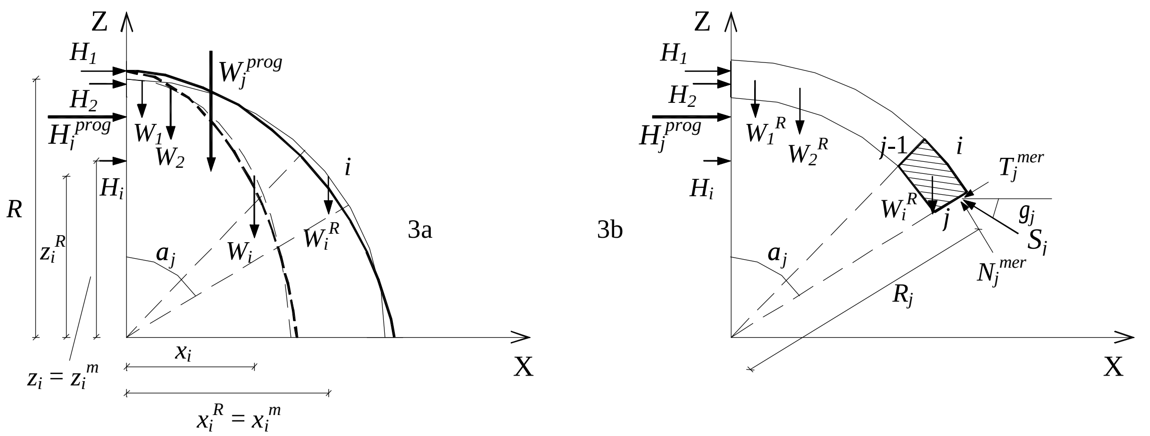
$$x_i = \frac{2}{p} x_i^R \qquad \text{for } i = 1 \text{ to } n \tag{4}$$

Said j a contact surface, the weight of the portion of half dome identified by angle a_j is:

$$W_j^{prog} = w p\, R^2 \left(1 - \cos a_j\right) \qquad \text{for } j = 0 \text{ to } n \text{ from the apex} \tag{5}$$

where *w* is the weight for unit surface of the mean domical surface.

The resultant of stresses in the parallel direction for each course is denoted in Fig. 3a by H_i, whose value depends on the position of the z_i. Given the isostatic nature of the problem, for any given set of z_i’s the state of stress is completely defined by simple equilibrium equations as seen in (1). The admissible values of H_i’s are limited by friction and this limitation yields geometries of the membrane surfaces that generally differ from the hemispherical one. The unique solution of the minimisation problem is defined by identifying the points of application of H_i’s for which the correspondent membrane surface yields the minimum thickness required, while satisfying the frictional constraints. For the minimisation problem z_i’s are assumed as the geometric unknowns and the geometry of the membrane surface is found with a discrete approach through the calculation of its coordinates *x* and *z* at a number of points, for instance one for each course.



(a) (b)

Figure 3: (a) Projection of the middle dome surface and hypothetical membrane on XZ. (b) : Projection of the generic block and unit meridian stress on XZ.

The coordinates of the generic membrane surface, whose projection on the plane XZ is represented with the thick curved continuous line in Fig. 3a, are:

$$x_i^m = x_i^R; \qquad z_i^m = z_i \tag{6}$$

while incremental and global equilibrium of each course yields:

$$H_j^{prog} = \frac{\Delta x_{i+1}}{\Delta z_{i+1}^m} W_j^{prog} \tag{7}$$

where: $\Delta x_{i+1} = x_{i+1} - x_i$; $\Delta z_{i+1}^m = z_i^m - z_{i+1}^m$
(8)

For each course the hoop stress is:

$$H_i = H_j^{prog} - H_{j-1}^{prog} \tag{9}$$

and the line of coordinates x_i (defined by (4)) and z_i, representing the locus of the points of application of the membrane stress resultants for the half dome, is drawn in Fig. 3a with a dashed thick line.

The unit meridian stress resultant S_j, applied to the generic interface j in Fig. 3b, forms with the X axis an angle g_j in general different from the angle a_j which identifies the inclination of the normal to the interface with the horizontal:

$$g_j = \arctan \frac{\Delta z_{i+1}^m}{\Delta x_{i+1}^m} \tag{10}$$

where $\Delta x^m_{i+1} = x^m_{i+1} - x^m_i$ defined by (6).

The membrane surface in (6) can then be described by its points on the interfaces by the following:

$$x_j^m = z_j^m \tan a_j; \qquad z_j^m = \frac{z_i^m + x_i^m \tan g_j}{1 + \tan a_j \tan g_j}$$

(11)

Relations (11) can be used to calculate the distance between the origin of the axes and each point of the membrane surface:

$$R_j^m = \sqrt{\left(x_j^m\right)^2 + \left(z_j^m\right)^2} \tag{12}$$

and hence the minimum constant thickness t required, is found by the relation:

$$t = \max \left| R - R_j^m \right| \tag{13}$$

which will be the objective function to minimise.

The material constraints are defined by limiting the maximum values of the internal shear force to be not greater than the frictional strength at each block interface. This limitation can be imposed independently on the two contributions obtained by the meridian and hoop stresses.

The meridian stress resultant for unit length of parallel at each interface j is S_j:

$$S_j = \frac{W_j^{prog}}{p\, x_j^m \sin g_j} \tag{14}$$

whose components normal and radially tangential to the interface are:

$$N_j^{mer} = S_j \cos\left(a_j - g_j\right); \qquad T_j^{mer} = S_j \sin\left(a_j - g_j\right) \tag{15}$$

Said m the friction coefficient, the limitation imposed by the failure criterion for the meridian stresses yields:

$$\left| T_j^{mer} \right| \le N_j^{mer}\, m \tag{16}$$

In order to define the frictional strength in the parallel direction for each course, let us now consider a single block of the course i+1, as part of a lune of dome identified by the horizontal angle q and the corresponding angle on the radial plane $q_j = q \sin a_j$ (Fig. 4). The latter angle and the length of the block in the parallel direction are variable with the vertical position of the block while the height of the block is kept constant. The dimensions of the block are:

$$a_j = q\, R \sin a_j; \qquad b = R\left(a_{j+1} - a_j\right) \tag{17}$$

Blocks belonging to adjacent courses, overlap each other for a length of the interface parallel dimension $c_j = a_j/2$, while:

$$c_j^m = c_j \frac{R_j^m}{R} = \frac{a_j^m}{2} \tag{18}$$

is the projection of the membrane surface parallel dimension on the interface plane.

The two components in the direction normal and tangent to the sliding interface of the meridian resultant of stresses ($S_j\, c^m{}_j$) can be obtained as follows (see Fig. 4):

$$\begin{aligned} N_{cj} &= S_j c_j^m \cos(a_j - g_j) \\ T_{cj} &= S_j c_j^m \sin(a_j - g_j) \end{aligned} \tag{19}$$

On the other hand the frictional strength of the j^{th} interface in the parallel direction is provided by the maximum value of T_{Pj} allowed by friction. As already shown in the previous section, the two normal and shear force resultants acting at this interface can be obtained by summing the two contributions of the meridian and hoop stresses, as follows:

$$\begin{aligned} N_{aj} &= 2\,N_{cj} - 2T_{Pj} \sin\frac{q}{2} \cos a_j \\ T_{aj} &= 2T_{Pj} \sin\frac{q}{2} \sin a_j - 2T_{cj} \cos\frac{q_j}{4} \end{aligned} \tag{20}$$

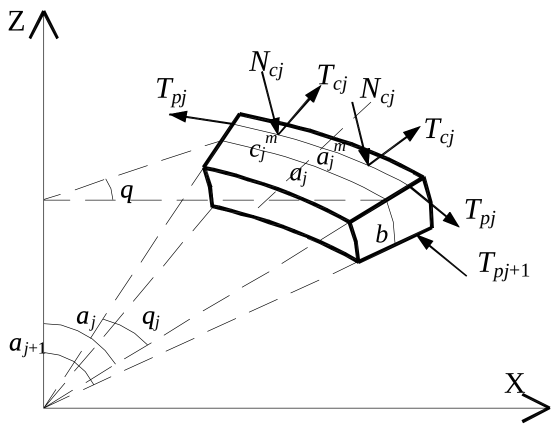


Figure 4 : Single block with contact forces.

and taking into account that: $T_{aj}^{\max} = N_{aj}\, m$ (21)

the system (20) yields:

$$T_{Pj}^{\max} = \frac{N_{cj}\, m + T_{cj} \cos(q_j / 4)}{\sin(q/2)\left(m \cos a_j + \sin a_j\right)} \tag{22}$$

so that the maximum tensile stress resultant that can be equilibrated in the circumferential direction is:

$$H_{i+1}^{\max} = T_{Pj+1}^{\max} - T_{Pj}^{\max} \tag{23}$$

Hence the frictional constraint on the hoop stress resultant is:

$$H_i \le H_i^{max} \tag{24}$$

applicable to the portion of dome where tensile hoop stresses arise.

Summarising, the optimum programming problem is completely defined as follows:

$$\text{Minimise } t = \max\left|R - R_j^m\right| \tag{25}$$

under the constraints:

$$\begin{aligned} &\left|T_j^{mer}\right| \le N_j^{mer}\, m \\ &H_i \le H_i^{max} \qquad \text{(for tensile } H_i\text{)} \end{aligned} \tag{26}$$

3 DISCUSSION OF RESULTS

The procedure described above has been used to derive the minimum constant thickness of a dome under self-weight.

The diagram in Fig. 5 shows the influence of the friction coefficient on the minimum thickness required. The results obtained with the approach described above are compared with the results obtained with the same procedure but following the assumption of Heyman's model of a dome already cracked along the meridians. It is worth noticing that Heyman's model (line "arch" in Fig. 5), yields a constant value of the *t/R* ratio of 0.0425 if the friction coefficient m is greater than 0.25. For $m < 0.25$, the resultant of stresses reaches a limit value on some block interface, according to equation (16) and the minimum thickness increases sharply. For values of $m < 0.2$ the thickness tends to infinity and the solver cannot find a feasible solution.

The model "membrane" differs from Heyman's model mainly by the fact that takes into account the structural capacity associated with the hoop stresses generated by the horizontal curvature of the dome. This is however limited by friction and the line "membrane" in Fig.5 clearly shows the increment in capacity associated with it. Specifically, for values of $m \ge 0.7$, the minimum thickness required is infinitely small as the membrane surface coincides with the mean hemispherical surface of the dome. As m decreases, for values $0.7 > m > 0.12$, the minimum thickness increases but remains smaller than the one required by Heyman's model. The variation in thickness in this range is associated with membrane surfaces that tend to move inward with respect to the middle hemispherical surface so as to accommodate for reduced admissible tensile action in the parallels. For values of $m < 0.08$ the thickness required increases sharply.

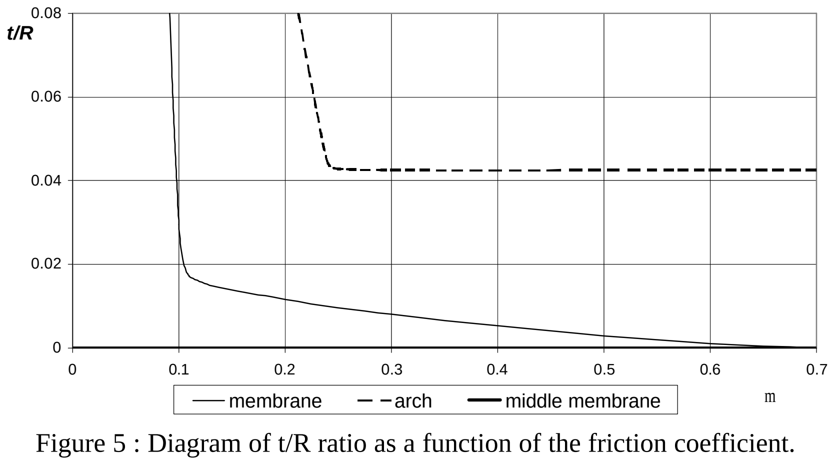


Figure 5 : Diagram of t/R ratio as a function of the friction coefficient.

The classic membrane, with no restriction on tensile stresses, represents the lower bound of possible solutions. This, independent from the friction coefficient, is represented in Fig.5 by a horizontal line which, in the specific case of no limitation on the value of meridian stresses, has equation $t/R = 0$ (called "middle membrane" in Fig.5).

The curve "membrane" is obtained by varying only the z co-ordinate of the representative points. A further assumption is that the value of parallel stresses at the support (horizontal surface) and hence resultant thrust will be absorbed by the external support. However in many real cases masonry domes sit on masonry drums with vertical walls and not greater thickness than the dome itself. Hence the procedure has also been used first relaxing the last condition, this yields the curve "vertical support" in Fig. 6; then also allowing for the x co-ordinate of the surface (equation (8)), to vary together with the z co-ordinate, this yields line "x+z variable" in Fig. 6.

For these two cases the limitation used is:

$$\begin{aligned} &\left|T_{mer}\right| \le N_{mer}\, m && \text{for compressive } H_i \\ &\sqrt{(H_i^{\,2} + T_{mer}^2)} \le N_{mer}\, m && \text{for tensile } H_i \end{aligned} \tag{27}$$

As it can be noted the line "vertical support" is the most sensitive to variation of the friction coefficient. Most importantly for value of $m > 0.35$, the required minimum thickness is still smaller than the one required by the Heyman's model ("cracked" in Fig. 6). However for lower values of m the required thickness increases quickly and there are no feasible solutions for values of the friction coefficient lower than about 0.26. This results highlights first that for materials with low friction coefficient there is a need for hoop chains at the support of the dome and this is the only way of absorbing the thrust, but also that it exist a range of values of the friction coefficient for which the Heyman model is not necessarily safe in terms of the thickness required. This can also be interpreted as meaning that the cracked state is a more stable condition for domes made of masonry with low friction.

It can however be argued that either buttresses or a drum with greater cross section than the dome are likely to be present at the base of the dome itself, and these will allow for the line of thrust to have the x co-ordinate different from the one of the middle surface for z = 0. This is obtained by allowing the x co-ordinate to change in the solution of the optimum problem. This yields a programming problem with a higher level of non linearity. The results obtained are shown by the curve "x+z variable" in Fig. 6. For high values of the friction coefficient this provides slightly improved solutions with respect to the "membrane" curve. However, for values of $m < 0.3$, this model offers more conservative values, and this is due to the inclusion of the support conditions. For values of $m < 0.136$ there are no feasible solutions.

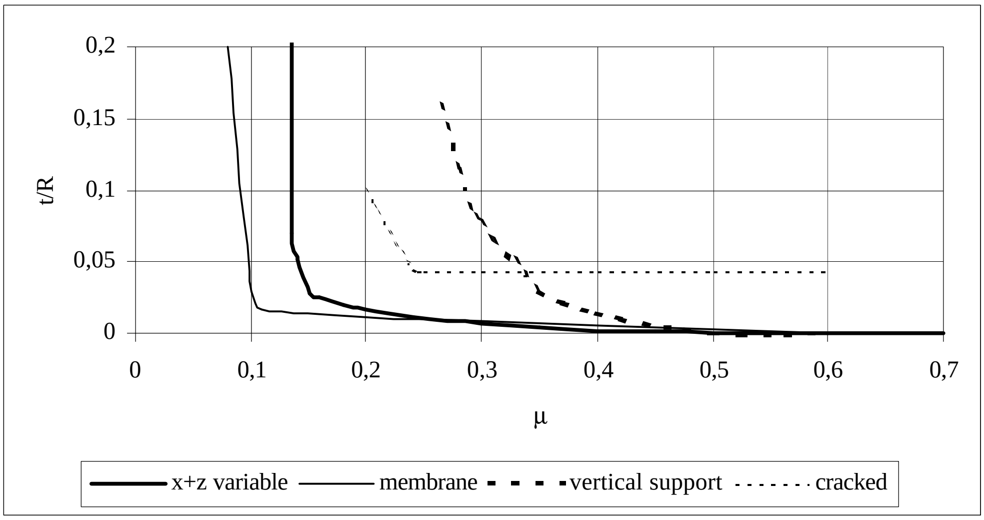


Figure 6 : Minimum thickness required taking into account support conditions.

4 CONCLUSIONS

The paper presents a strategy for proving the uniqueness of the solution of the limit state analysis of block masonry domes governed by finite friction and subjected to self-weight load.

Using a lower bound approach, a linear programming strategy is developed to define the minimum thickness required for the dome to maintain its stability. The results are compared with the classic Heyman's model. Different conditions of boundary are also discussed and it is proved that the assumption of vertical crack and arch behaviour is not necessarily the safest for values of the friction coefficient smaller than 0.35.

In the previous analysis no restriction is based to the maximum value of stress in the meridian and indeed the assumption of the membrane surface coinciding with the external or internal edge of the dome implies infinite stresses. Therefore values of the required minimum thickness, which will compare better with real situations, can be obtained from the previous analysis, if it is assumed that the calculated thickness coincides with the medium third of the physical cross section of the dome.

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REFERENCES

Baggio, C. and Trovalusci, P. 1998a. Mathematical programming application to limit analysis of three-dimensional masonry structures. *Proc. 5th Int. Masonry Conf.*, 8, p. 131-134.

Baggio, C. and Trovalusci, P. 1998b. Limit analysis for no-tension and frictional three-dimensional discrete systems. *Mech. Struct. & Mach*. 26(3), p. 287-304.

Casapulla, C. 1999. Problematiche relative all'analisi limite di strutture murarie a blocchi in presenza di attrito. *Ph.D. Thesis*, Dept. of "Costruzioni", Univ. of Firenze.

Casapulla, C., De Riggi, T. and Jossa, P. 1998. Cinematica di blocchi affiancati con attrito in presenza di azioni sismiche. *Ingegneria Sismica* 3, p. 21-34.

Casapulla, C. and Jossa, P. 2001. A safety method in static analysis of block masonry. In press in *Structural Analysis of Historical Constructions*; *Proc. III intern. seminar, Guimaraes, 7-9 November 2001*.

Casapulla, C. and D'Ayala, D. 2001. Lower bound approach to the limit analysis of 3D vaulted block masonry structures. In press in *Computer Methods in Structural Masonry (STRUMAS V);Proc. 5th intern. symp., Roma, 18-20 April 2001*.

D'Ayala, D. 1993. Analytical method for the assessment of the safety levels of domes. In *Structural Studies, Repairs and Maintenance of Historical Buildings (STREMAH)*; *Proc. 3th intern. conf., Bath, 1993*, p. 347-355.

D'Ayala, D. 1994. In tema di comportamento strutturale delle cupole in muratura. *Ph.D. Thesis*, Dept. of "Ing. Strutt. e Geot.", Univ. "La Sapienza", Roma.

Heyman, J. 1977. *The equilibrium of shell structures.* Oxford: Clarendon Press.

Livesley, R.K. 1978. Limit analysis of structures formed from rigid blocks. *Int. Journ. for Num. Meth. in Eng*. 12, p. 1853-1871.

Livesley, R.K. 1992. A computational model for the limit analysis of three-dimensional masonry structures. *Mecc.* 27(3), p. 161-172.

Lo Bianco, M. and Mazzarella, C. 1983. Limit load of masonry structures. In *Final Report IABSE symp*., Venezia, p. 187-194.

Oppenheim, I.J, Gunaratnam, D.J. and Allen, R.H. 1989. Limit state analysis of masonry domes. *ASCE Journal of Structural Engineering* 115(4), p. 868-882.

O'Dwyer D. 1999. Funicular analysis of masonry vaults. *Computers and Structures* 73, p. 187-197.

Sinopoli, A., Corradi, M. and Foce, F. 1997. Modern formulation for preelastic theories on masonry arches. *ASCE Journal of Engineering Mechanics* 123(3), p. 204-213.

Villaggio, P. 2000. The slope of Roofs. *Meccanica* 35, p. 215-227.