A rigid plastic model of the under-excavation technique applied to stabilise leaning towers

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ABSTRACT: This paper aims at analysing the effect of under-excavation on the stabilisation of leaning towers. In accordance with some experimental results it discusses the existence of a critical extent of the under-excavated region beyond which the soil removal aggravates the initial tilt of leaning towers. A rigid plastic model is developed for the prediction of the critical extent in function of the principal parameters characterising the under-excavation process.

1 INTRODUCTION

The under-excavation technique (Terracina, 1962) that was successfully applied in Mexico (Tamez et al., 1995) has been adopted to stabilise the Tower of Pisa (National Committee of the Tower of Pisa, 1999). This technique consists in removing, by borings, small volumes of soil from the raised side of the leaning tower.

An experimental study carried out at the Imperial College of London (Edmunds, 1993) showed that the efficacy of the under-excavation intervention as a means to stabilise leaning towers is strongly affected by the extent and location of the soil region subjected to under-excavation. In particular, this study predicts a *critical extent* of the under-excavated ground region beyond which soil extraction aggravates the initial tilt of the tower but before which it provides a decrease in the tilt.

This experimental result represents an important issue for a successful application of the under-excavation process and, thus, requires to be modelled in order to predict the *safe zone* of under-excavation where the soil extraction causes a reduction of the tower's tilt.

The present paper aims to give a contribution to the theoretical modelling of this aspect. The central point of the study is that the experimental *critical extent* of the under-excavation region, causing the increasing of the initial tilt of the foundation, can be understood only by adopting a plastic approach for the response of the soil-foundation system.

At first the under-excavation process is associated with a reduction of the tower's tilt. In fact the direct extraction of the ground induces the formation of small cavities that collapse. Consequently, the excavation causes differential settlements beneath the raised side of the tower that reduce the initial tilt. In addition, according to the results of a previous study (Como et al., 2001), the elastic response of the soil-foundation system is always associated with a *counter-rotation* of the foundation in the opposite direction of the initial tilt. In other words, neither the elastic behaviour of the system nor the direct extraction of the ground provide a satisfactory explanation for the experimental results concerning the *critical extent* of the under-excavated region. As an alternative, when the extent of under-excavation is greater than the critical one, the authors suggest the hypothesis that the increase of the tower's tilt is related to the plastic response of the soil-foundation system.

This paper will discuss this hypothesis in detail. The under-excavation process will be analysed in the framework of the plastic response of the soil- foundation system. Particularly, a model for a rectangular foundation placed on a rigid plastic soil will be developed and compared with some experimental results reported in the literature.

2 PLASTIC ANALYSIS OF THE UNDER-EXCAVATION PROCESS

2.1 The limit interaction locus of an eccentrically loaded foundation: some remarks

Let us consider a rigid foundation loaded by a vertical load N and a moment M and assume a global plastic model for representing the mechanical behaviour of the soil-foundation system. In the plane (N, M), the failure locus of the foundation is represented by the interaction locus Y shown in Figure 1. In the same figure the point P(N,M) represents the generalized stress state acting on the foundation.

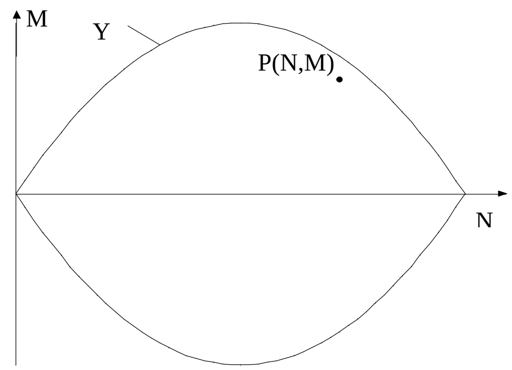


Figure 1: the failure locus.

In particular, for a rigid rectangular shape foundation having the dimensions equal to B and L and/or a strip shape foundation having the width equal to B, the model proposed by Meyerhof, (1953) provides a limit load equal to that of an equivalent foundation subjected to a centred load and characterised by a smaller width given by:

$$B' = B - 2e \tag{1}$$

where e=M/N is the eccentricity of N with respect to the foundation's centroid.

The relationship between the vertical load N and the moment M=Ne is described by the following equilibrium condition:

$$M = N\,\frac{B}{2}\left(1 - \frac{N}{N_0}\right) \tag{2}$$

where N_0 represents the limit load of the centred loaded foundation given by:

$$N_0 = \sigma_0\; B\,L \tag{3}$$

being σ_0 the limit soil pressure acting at the contact surface between the foundation and the soil and represented in Figure 2c.

Figure 2 shows a variety of plastic states activated in function of the entity of N and M and localised on the interaction locus corresponding to N_0.

2.2 The effect of the under-excavation on the response of the soil-foundation system

Let us now consider the beginning of an under-excavation intervention and suppose that the ground extraction occurs through the soil volume involved in the plastic collapse mechanism of the foundation. The starting point of the study consists of assuming that the under-excavation process induces an increase in soil porosity with a subsequent reduction of strength in the region subjected to the ground extraction. This implies that the soil beneath the raised side of the foundation assumes physical and mechanical characteristics, such as density, friction angle and cohesion, which are different from those characterising the leaning side of the foundation.

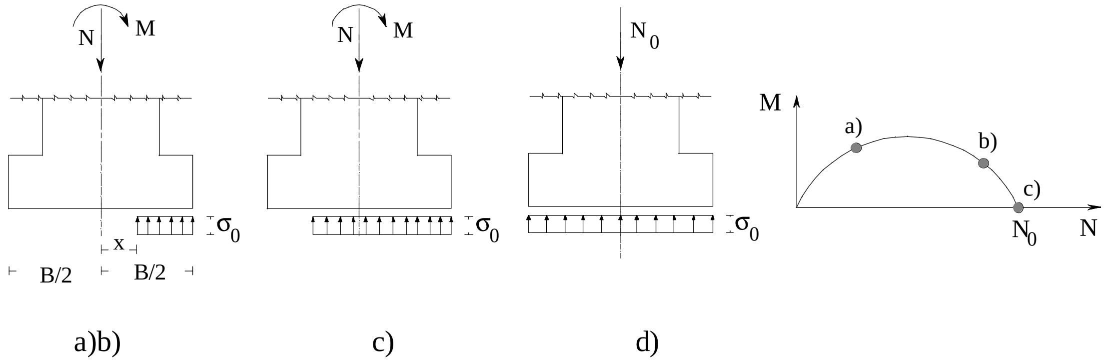


Figure 2: Plastic states of the eccentrically loaded foundation.

We assume that, as the plastic collapse condition is reached, the distribution of the limit interaction stresses is not anymore uniform. This is shown in Figure 3 where the smaller limit soil pressure $\beta\sigma_0$ ($0 \le \beta \le 1$) is assumed to act at the contact surface between the foundation and the soil for the extent αB ($0 \le \alpha \le 1$) of the under-excavation region whereas the initial limit soil pressure σ_0 still acts on the remaining region of extent equal to $(1-\alpha)B$.

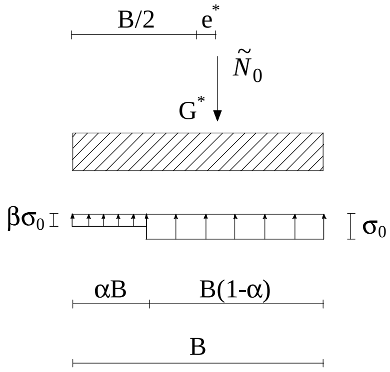


Figure 3: Limit soil pressure distribution at the contact surface.

Figure 3 also illustrates the modified strength centroid, G*, placed on the leaning side of the foundation at the distance e* from the centre of the foundation.

On the basis of the above assumption, the limit load, $\tilde{N}_0$, of the *under-excavated foundation* applied at the strength centroid G* is less than the limit load, N_0, of the original foundation and is given by:

$$\tilde{N}_0 = N_0 \left[1 - \alpha(1-\beta)\right] \tag{4}$$

The distance, e^*, of G* from the centre of the foundation is given by:

$$e^* = \frac{\alpha B}{2}\left(1 - \frac{\beta}{1-\alpha(1-\beta)}\right) \tag{5}$$

The expression (5) gives e*=0 either in the case of absence of under-excavation (β=1) or in the case of a uniform under-excavation process (α=1). Otherwise it results that $e^* \ge 0$.

It is evident that the values of α and β should be adequately defined in function of the under-excavation process.

If we assume that the value of α is less than 0.5, the modification of the soil characteristics caused by the under-excavation has a double effect. First, the eccentricity of the vertical load with respect to the instantaneous strength centroid decreases in dependence of the different entity of the under-excavation process. This means that, with reference to Figure 4, the point P(N,M), representative of the stress state acting on the foundation referred to the point G*, moves parallel to the M-axis towards the interior of the interaction limit locus Y assuming the different positions P_1, P_2, P*. Second, the interaction limit locus Y shrinks in function of the reduction of the limit soil pressure in the zone subjected to the under-excavation process turning into the modified domains Y_1, , Y_2, Y*.

Then the increase in the initial foundation's tilt is associated with the fact that during the under-excavation process the stress state acting on the foundation changes its position until it reaches the boundary of an interaction locus characterized by positive values of M. This is shown in Figure 4 when the point P(N,M), reaches the position P* belonging to the boundary of the corresponding shrunk limit loci Y*. This produces a plastic strain increment $\dot{\varepsilon}$ characterised by an increase of the initial tilt in the direction of the slope side of the foundation.

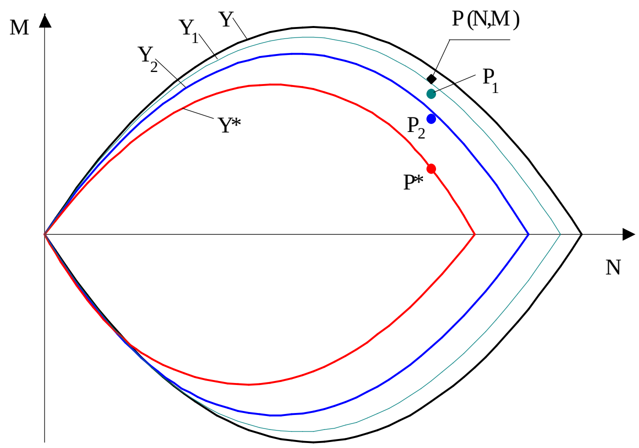


Figure 4: The effect on the under-excavation on the response of the soil-foundation system.

In the case of an under-excavation process with α, variable the value of e* reduces when $\alpha = \sqrt{\beta} - 1/\beta - 1$ and the point representing the stress state begins to move upwards.

2.2.1 The limit interaction locus for the under-excavated foundation

Let us now consider the vertical load N and the moment M acting at the point G* and derive the limit interaction locus modified by the under-excavation.

Figure 5 illustrates the different positions that the neutral axis can assume in the limit condition. Figure 5a represents the condition of positive moments with $0 \le N \le (1-\alpha)N_0$. In this case the equilibrium conditions provide:

$$\begin{aligned} N &= L\sigma_0 x \\ M &= L\sigma_0 x\left(\frac{B-x}{2} - e^*\right) \end{aligned} \tag{6}$$

The first of the above equations provides x which can be substituted in the second one giving the following non-dimensional expression:

$$m(n) = 4n\left[(1-n) - \alpha\left(1 - \frac{\beta}{1-\alpha+\alpha\beta}\right)\right] \qquad 0 \le n \le 1-\alpha \tag{7}$$

being:

$$n = \frac{N}{N_0} \qquad m = \frac{8M}{N_0 B} \tag{8}$$

Figure 5b represents the condition of positive moments with $(1-\alpha)N_0 \le N \le (1-\alpha+\beta\alpha)N_0$. The equilibrium equations are expressed as:

$$\begin{aligned} N &= L\sigma_0\left[B(1-\alpha)(1+\beta)+\beta x\right] \\ M &= L\sigma_0\left\{\alpha B^2\frac{(1-\alpha)}{2} - \beta\left[B(\alpha-1)+x\right]\left(\frac{x-\alpha B}{2}\right)\right\} - Ne^* \end{aligned} \tag{9}$$

The non-dimensional form can be written as:

$$m(n) = \frac{4}{\beta}\left\{n\left[\frac{(\beta-1)\left[\alpha^2(\beta-2)+(4\alpha-1)\right]+1}{(1-\alpha+\alpha\beta)} - n\right] + (\alpha-1)^2(\beta-1)\right\} \qquad 1-\alpha \le n \le 1-\alpha+\beta\alpha \tag{10}$$

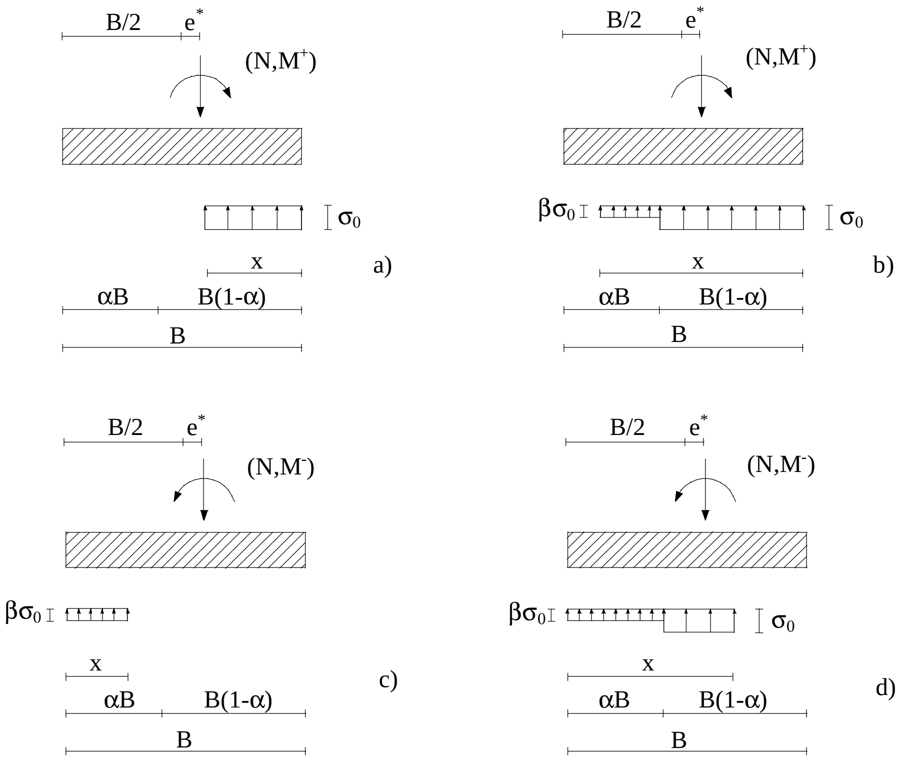


Figure 5: The position of the neutral axis.

The condition of negative moments with $0 \le N \le \alpha\beta N_0$ is illustrated in Figure 5c, where the neutral axis crosses the under-excavated region. In this case the equilibrium equations are:

$$\begin{aligned} N &= \beta L\sigma_0 x \\ M &= -\beta L\sigma_0 x\left(\frac{B-x}{2} + e^*\right) \end{aligned} \tag{11}$$

which can be expressed by the following non-dimensional form:

$$m(n) = 4n\left(\frac{n}{\beta} - \frac{1-\alpha^2+\alpha^2\beta}{1-\alpha+\alpha\beta}\right) \qquad 0 \le n \le \alpha\beta \tag{12}$$

The condition of negative moments with $\alpha\beta N_0 \le N \le \tilde{N}_0$ is shown in Figure 5d, where the neutral axis is external to the under-excavated region. The equilibrium conditions provide:

$$N = \beta N_0 \left[\alpha(1-\beta) + \frac{x}{B} \right]$$
$$M = -L\sigma_0 \left[\alpha\beta B^2 \frac{(1-\alpha)}{2} + (x - \alpha B)\left(\frac{B(1-\alpha) - x}{2} \right) \right] - Ne^* \tag{13}$$

The non-dimensional form can be expressed as follows:

$$m(n) = 4n \left\{ n + \left[\frac{\alpha^2\beta - \alpha^2 + 2\alpha - 1}{1 - \alpha + \alpha\beta} - 2\alpha\beta \right] \right\} - (1-\beta)\alpha^2\beta \qquad \alpha\beta \le n \le 1 - \alpha + \alpha\beta \tag{14}$$

The previous results are summarised in Figure 6 where is illustrated the non-dimensional interaction limit loci obtained for α=0.4 and β=0, 0.25, 0.5, 0.75, 1 that is, for a fixed extent of the under-excavated region and increasing values of the entity of under-excavation intervention. Obviously, the case β=1 represents the absence of the under-excavation, that is, the case of the unmodified foundation.

3 THE CRITICAL REGION OF THE INITIAL LIMIT INTERACTION LOCUS

In the non-dimensional plane (n, m) the stress state acting on the foundation is represented by the point ($\overline{n}, \overline{m}$) being $\overline{m}$ the moment evaluated with respect to the geometrical centroid of the foundation. By referring the moment to the effective strength centroid G*, the stress state, named ($\hat{n}, \hat{m}$), is defined as follows:

$$\hat{n} = \overline{n}$$
$$\hat{m} = \overline{m} - \overline{n} \cdot d(\alpha, \beta) \tag{15}$$

being d(α,β) the value of e* normalized with respect to B/8 which is given by:

$$d(\alpha, \beta) = \frac{8e^*}{B} = 4\alpha\left(1 - \frac{\beta}{1 - \alpha(1-\beta)} \right) \tag{16}$$

Now, let us analyse the case of assuming a constant value of α and reducing the value of β from one to zero.

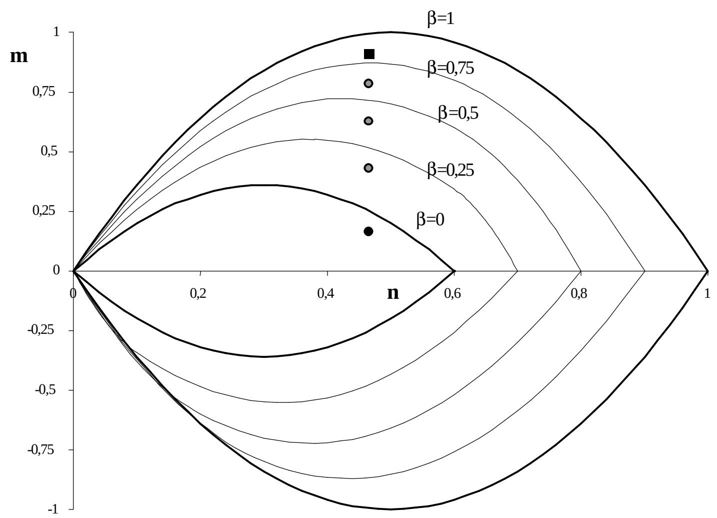


Figure 6: Case 1. The under-excavation is not associated to plastic strains.

On the basis of the previous considerations the under-excavation process induces a reduction of $d(\alpha,\beta)$. As a consequence, from the second of (15) the moment $\hat{m}$ gradually decreases and the point $(\hat{n},\hat{m})$ moves towards the interior of the domain in the direction of the m-axis.

It is evident that the response of the *under-excavated foundation* is affected by the values of α and β characterising the under-excavation process and by the initial stress state $(\bar{n},\bar{m})$ acting on the foundation. In this case we can have three modes of response:

1 The under-excavation does not cause any plastic rotation increment of the foundation. This mode is illustrated in Figure 6 where are reported the interaction limit loci and the corresponding stress state points obtained by fixing $\alpha = 0.4$ and $\beta = 0, 0.25, 0.5, 0.75, 1$. The initial stress state point $(\hat{n},\hat{m})$, corresponding to $\beta = 1$ and represented by the solid box, remains internal to the corresponding interaction locus. Thus no values of β causes plastic strain increments and the under-excavation process is safe.

2 The point $(\hat{n},\hat{m})$ touches the boundary of the interaction limit locus in the region characterised by negative moments. Figure 7 illustrates the interaction limit loci and the corresponding stress state points obtained by fixing $\alpha = 0.4$ and $\beta = 0.5, 0.75, 1$. At $\beta = 0.5$ the stress state point touches the boundary of the corresponding locus and any reductions of β produces a plastic strain increment $\dot{\varepsilon}$ normal to the locus boundary. The vertical component of $\dot{\varepsilon}$ is negative, which means that the plastic rotation increment occurs in the opposite direction of the initial moment with a consequent reduction of the initial foundation's tilt.

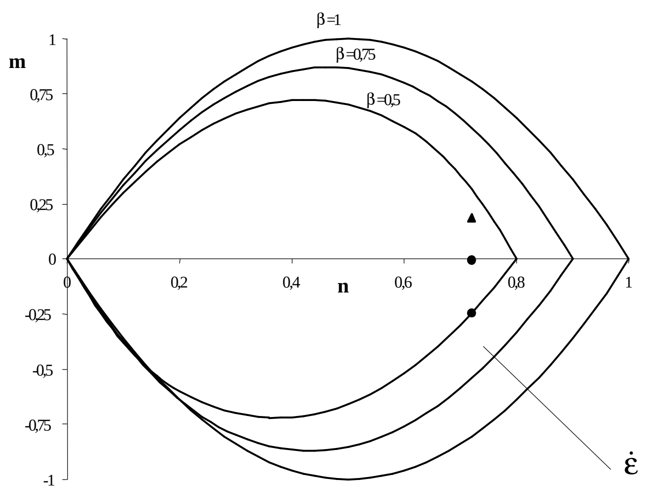


Figure 7: Case 2. The under-excavation is associated to a tilt reduction.

3 The point $(\hat{n},\hat{m})$ touches the boundary of the interaction limit locus in the region characterised by positive moments. Figure 8 illustrates the interaction limit loci and the corresponding stress state points obtained by fixing α? 0.4 and $\beta = 0.5, 0.75, 1$. At $\beta = 0.5$ the stress state point touches the boundary of the corresponding locus and any reductions of β produces a plastic strain increment $\dot{\varepsilon}$ normal to the locus boundary. In this case the vertical component of $\dot{\varepsilon}$ is positive, which means that the plastic rotation increment occurs in the same direction of the initial moment with a consequent increasing in the initial foundation's tilt.

Similar results can be obtained during an under-excavation process characterised by decreasing values of α and a constant value of β.

Given the type of the under-excavation process, characterised by the constant value of α (or β) and by the final value of β (or α), the occurrence of the above circumstances is clearly dependent on the initial stress state $(\bar{n},\bar{m})$.Thus it is worth defining a curve, named *critical line*, which separates the region of the initial interaction domain where lie the stress state points which do not allow the development of plastic strains associated to an increase of tilting (case 1 and 2) from the region where the risk of upsetting exists (case 3). The *critical line* is defined by the following equation:

$$\hat{m} = \overline{m} - \overline{n}\alpha\left(1 - \frac{\beta}{1-\alpha(1-\beta)}\right) = 0 \qquad \forall\alpha,\beta \tag{17}$$

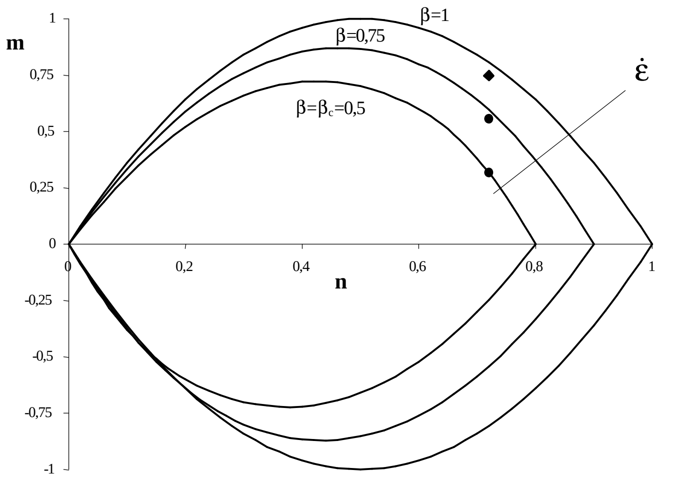


Figure 8: Case 3. The under-excavation is associated to a tilt increasing.

Let us consider an under-excavation process with uniform values of β with α varying between 0 and 1. Now considering the fact that on the critical line it results:

$$\overline{n} = 1 - \alpha + \alpha\beta \tag{18}$$

we can derive α from (18), substitute it into the equation (17) and obtain:

$$\overline{m} = \frac{(1-\overline{n})(\overline{n}-\beta)}{1-\beta} \tag{19}$$

The expression (19) represents the parabola shown by the dashed line in Figure 9a.
The region comprised between the parabola (19), the boundary of the initial interaction limit locus (α=0) and the interaction curve referred to α=1 represents the *critical region*.
In the case of an under-excavation process with uniform values of α, β can be still derived from (18) and substituted into the equation (17) obtaining:

$$\overline{m}(\overline{n}) = (1-\overline{n})(1-\alpha) \tag{20}$$

The expression (20) represents the dashed line shown in Figure 9b. The region included between the boundary of the initial interaction limit locus (β=1) and the critical line represents the critical region. The same figure shows the points representative of the initial stress states referred to Figures 6, 7, 8.

If the point, representing the initial stress state ($\overline{n}, \overline{m}$), is inside the critical region it is possible to evaluate the values of α and β, named α_c e β_c, which cause the contact between the point and the boundary of the interaction limit locus.

The critical values are given by the following equation:

$$m(\hat{n}) = \hat{m} \tag{21}$$

where $m(\hat{n})$ is the expression (10) evaluated in $\hat{n}$.

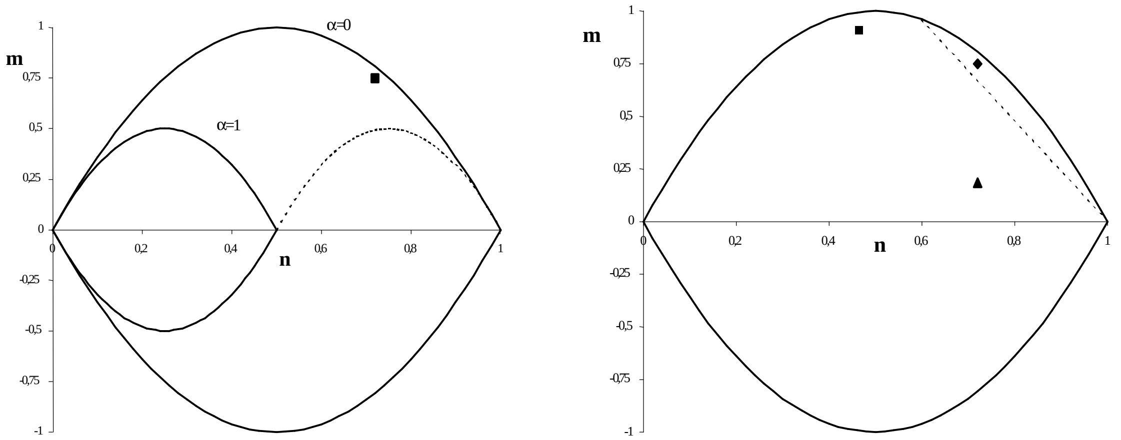
By omitting the intermediate calculations we have:

$$\beta_c = \frac{(\overline{n} - 1 + \alpha)^2}{\alpha^2 + (1-\overline{n})(1-2\alpha) - \overline{m}} \tag{22}$$

in the case of an under-excavation process characterised by uniform values of α, and

$$\alpha_c = (1 - \bar{n}) + \sqrt{\frac{\beta\left(\bar{n} - \bar{n}^2 - \bar{m}\right)}{1 - \beta}} \tag{23}$$

in the case of an under-excavation process characterised by uniform values of β.



a) b)

Figure 9: The critical line a) under-excavation process with β=const. b) under-excavation process with α=const.

4 NUMERICAL RESULTS

The rigid plastic model proposed in the above sections is applied to a square shaped foundation with the side length equal to 102 mm. The foundation is loaded by a vertical load equal to 165 N and a moment equal to 2300 $N \cdot mm$. The side of the foundation is chosen so that the square shape foundation inscribes the circle shape foundation analysed by (Edmunds, 1993) in the experimental investigation. Moreover the stress state (N,M) is the same as that induced on the experimental tower by its initial tilt equal to 5.5°. The soil limit pressure is equal to $0.022\ N \cdot mm^{-2}$ assumed on the basis of a zero cohesion and a friction angle equal to 35°. The proposed model is applied assuming an under-excavation process with the fixed value of α equal to 0.32 which represents the *critical value* obtained by Edmunds during one of the performed experimental investigations.

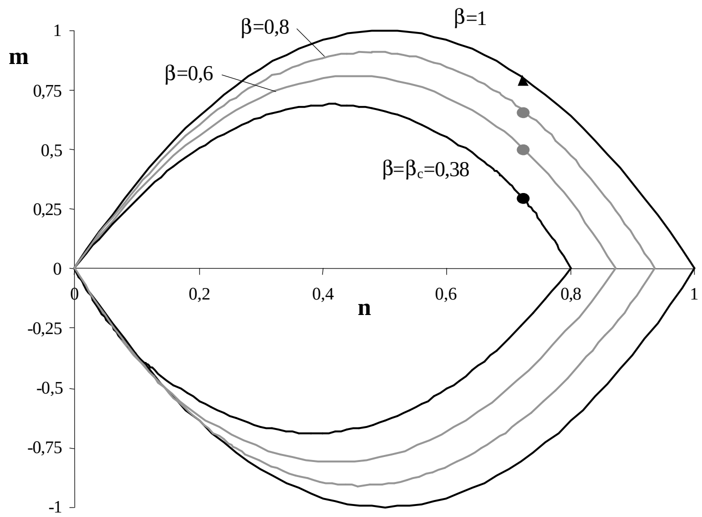


Figure 10: A numerical example.

The results of the proposed model are reported in Figure 10, which illustrates the limit interaction loci obtained for different values of β.

The point is inside the unsafe region of the initial domain. Indeed, for $\beta=\beta_c=0.38$ the point (N,M) reaches the boundary of the corresponding limit interaction locus. At this point any further reduction of β produces a plastic strain increment normal to the locus boundary that shows a positive vertical component. In other words, a plastic rotation increment occurs and the initial foundation's tilt increases. This means that the model provides a value of β such that $0 \le \beta_c \le 1$ in correspondence of which the value of α=0.32 is critical. The results shown in Figure 10 represent an application of the proposed model that is consistent with the experimental investigation carried out by Edmunds. It is worth underlying that in this application we do not provide any proposal concerning the correlation between the coefficient β of the model and the strength reduction of the under-excavated soil that would require further analyses also in the specific context of a geotechnical study.

4 CONCLUSIONS

In this paper we analysed the effect of the under-excavation technique on the stabilisation process of leaning towers. We developed a model for a rectangular foundation placed on a rigid plastic soil that is consistent with some experimental results reported in the literature. This model provided an efficient tool to predict the existence of a critical extent of the region of under-excavation beyond which the ground extraction aggravates the initial tilt of leaning towers. In the framework of this simplified rigid plastic model, we defines the equations of the critical line that separates the *safe region* of the initial interaction domain where lie the stress state points which do not allow the development of unsafe plastic strains from the *critical region* associated with an increase of the initial tilt. In the latter case we gave the analytical expression of the critical values α_c e β_c related to the condition of contact between the stress state point and the boundary of the interaction limit locus.

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