Limit analysis of the structures of Colosseum

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ABSTRACT: In this paper the analysis of the safety conditions of the *Colosseum* in Rome is performed with a special concern to the static loading conditions. The limit analysis approach together with the assumption of the no-tension constitutive model represents an effective tool in evaluating the safety conditions and the ultimate strength of masonry structures. In the context of this approach the static analysis of the structure of *Colosseum* is performed.

A study devoted to evaluate the collapse load acting on the vaults covering the inner and outer ambulatories is carried out. Both the circumferential and the radial arches are considered to be at a minimum thrust condition due to the radial outward displacements of the external and/or central walls. The study is referred to the original configuration of the construction.

1 INTRODUCTION

The analysis of the structure of Colosseum is quite complex because of the difficulties concerning the evaluation of loads and the definition of constitutive laws able to model the behavior of materials such as stones, tufa, concrete and masonry, which are characterized by low or zero tensile strengths and damage properties. Moreover the complex geometry of the Colosseum makes the modeling and the definition of simplified structural schemes quite difficult. Among the main approaches adopted to analyze the behavior of historical constructions, the limit analysis represents an effective tool for understanding the main aspects of the ultimate behaviour and evaluating the ultimate capacity of the constructions.

The study adopts a rigid no-tension constitutive model with no sliding (Heyman, 1977; Heyman, 1982; Como, 1992). This assumption, which was firstly introduced by (Heyman, 1977; Heyman, 1982), implies that the masonry body behaves as an assemblage of rigid elements kept together by compression forces and cracked at regions characterised by tensile stresses.

The no-tension constitutive assumption is certainly verified for the masonry made of rigid blocks with no mortar. However it can be applied also to ancient masonry made of tufa blocks or bricks with mortar joints or to the case of *opus caementicium* considering the fact that the mortar becomes weaker during the time and looses its initial tensile capacity (Giovannoni, 1994). The no-tension constitutive hypothesis can be adopted also for materials characterised by a finite tensile capacity if we consider the fact that the tensile capacity can be suddenly deteriorated because of some events such as dynamic actions which cause cracking.

The masonry constitutive model can be defined in details on the basis of the hypotheses introduced by (Heyman, 1977; Heyman, 1982) in the analysis of the strength of masonry structures and expressed through the following expressions:

$$\underline{\sigma} \leq 0 \tag{1}$$

$$\underline{\varepsilon}^{(f)} \geq 0 \tag{2}$$

$$\underline{\sigma} \cdot \underline{\varepsilon}^{(f)} = 0 \tag{3}$$

The condition (1), referred to the principal components of the stress tensor $\underline{\sigma}$, defines the domain Y of the admissible stresses and indicates the absence of tensile stresses. The condition (2), referred to the principal components of the cracking strain tensor $\underline{\varepsilon}^{(f)}$, defines the domain Y' of the admissible strains and indicates the absence of contractions. The normality rule (3) indicates that cracking can develop only at points and along directions where the compression stresses are zero. It expresses the absence of internal dissipation in correspondence of cracking states. Finally if P is a point of the masonry body, $\underline{n}$ is the outward normal at the surface where P belongs and $\underline{t}_n$ is the associated stress vector, the equation expressing the convexity condition for the limit surface is:

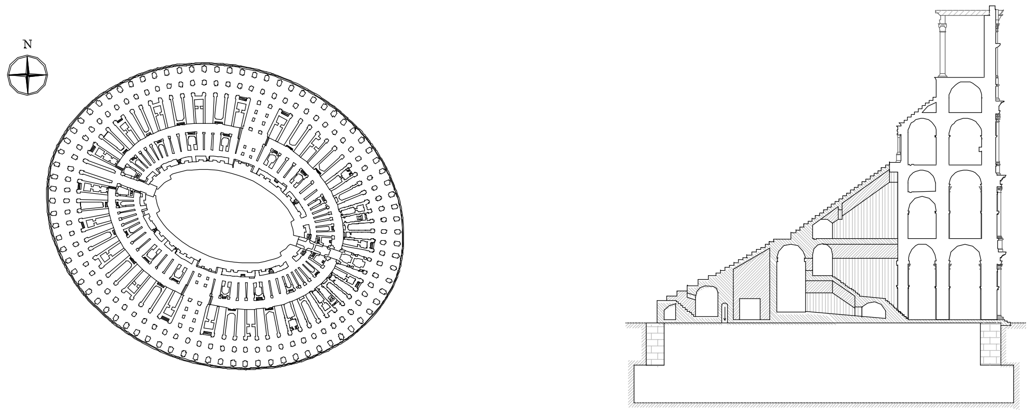
$$\underline{t}_n \cdot \underline{n} \leq 0 \qquad (4)$$

The convexity condition of the limit surface and the normality rule make it possible the extension of the limit analysis procedure to masonry structures.

In this context the present work discusses the results of a study concerning the structural safety of the Colosseum at vertical loading conditions and with reference to the original configuration of the construction. The study is based on the geometrical and material data reported in (Cerone, 2000) and is developed within the National Research Program "*Ricerche per il restauro e la valorizzazione dell'Anfiteatro Flavio"* which involves the three Universities of Rome and the Soprintendenza Archeologica di Roma.

2 THE KINEMATICS OF THE STRUCTURE UNDER VERTICAL LOADS

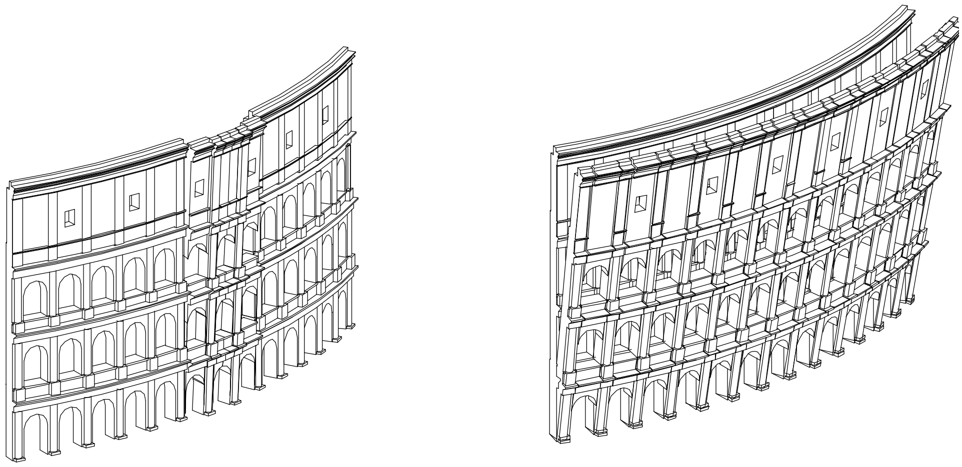
The original configuration of the Colosseum has the elliptical plan illustrated in Figure 1a with the external diameters equal to 188 m and 155 m. The main part of the structure is constituted by three ring walls with columns made of travertine blocks without mortar and connected by means of arches and *architrave* at various levels which support the ambulatory vaults. The external circumferential wall is characterised by the presence of a superior cornice, the *attico*, with forty rectangular windows. The inner circumferential wall is connected to radial masonry walls that are the support of the *cavea*. In Figure 1b is reported a radial section of the structure and it is shown the foundation of this part of the structure made by a large and massive continous concrete ring whose width is greater than 50 m.



a) b)

Figure 1: a) plan; b) radial section.

The structure is, hence, characterised by a high shape resistance which prevents single parts of the external wall from being overturned outwards due to the thrusts transmitted by both the arches and the ambulatory vaults (Figure 2a). This can be explained by the following reasons. First, the single parts of the external wall would not rotate because of the high friction resistance caused by the compactness of masonry and the compression transmitted by the arches. Second, each partial mechanism, involving only single parts of the walls, is not compatible with the material constitutive assumptions because it would cause a closure of the stone voussoirs and a consequent penetration of the materials. Thus a compatible mechanism should involve the rotation of the entire external wall outwards as it is shown in Figure 2b. In this case the wall is subjected to hoop stretches which cause vertical cracking.



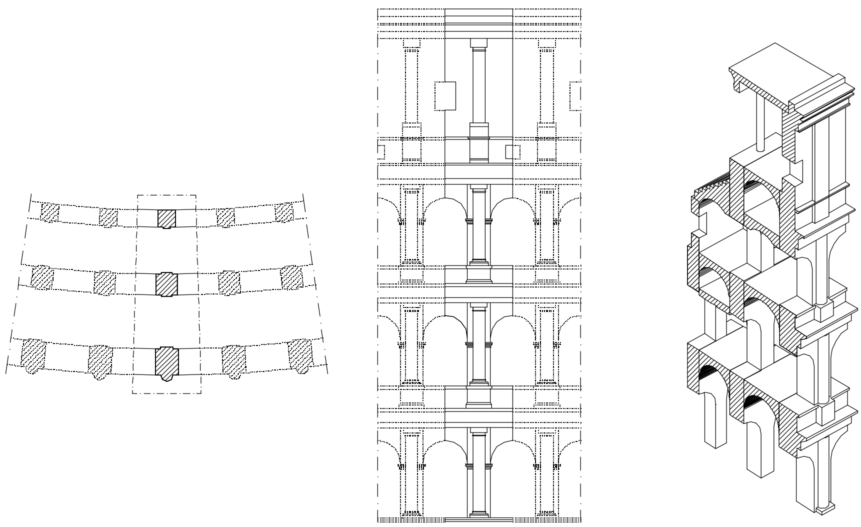
a) b)

Figure 2: The external wall: a) the incompatible mechanism; b) the compatible mechanism.

It is clear that the same consideration is still valid with reference to the central wall. On the contrary the internal wall is prevented to rotate inwards because of the constraint imposed by the radial walls. Then the possible mechanisms can be of two types:

- mechanism type a) involving the rotation of the external wall;
- mechanism type b) involving the rotation of both the external and the central walls.

According to the kinematic theorem, the collapse load is the minimum among those related to each mechanism. In the following sections the possible mechanisms of the structure will be analysed. In particular all the calculations will be referred to the module illustrated in Figure 3.



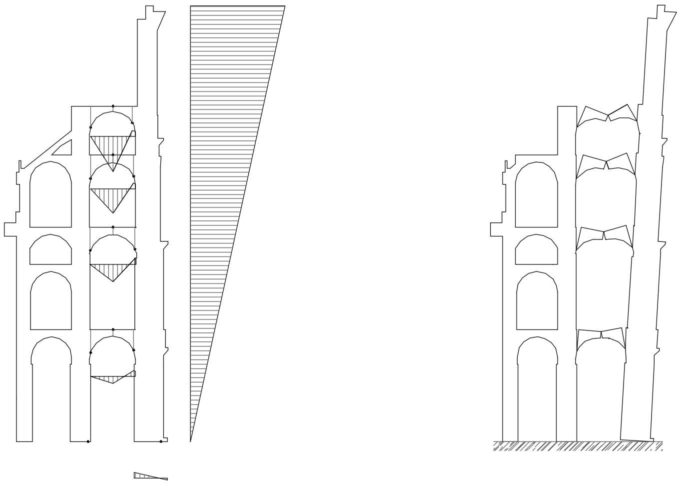
a) b) c)

Figure 3: Module of the structure: a) plan; b) front view; c) axonometry.

3 THE MECHANISMS OF THE RADIAL SECTION

3.1 The mechanism type a)

This type of mechanism involves the external wall and the external ambulatory vaults. The external wall rotates outwards while the inside and the central walls are fixed. The external vaults, which are included between the fixed central wall and the rotating external wall, crack and, for each one of them, three hinges will form, two at the intrados of the abutments and one at the extrados. Figure 4 shows a possible mechanism of the structure and indicates an arbitrary distribution of hinges. The same figure illustrates the vertical and horizontal components of the displacement field.



a) b)

Figure 4: a) vertical and horizontal displacements; b) radial section of the mechanism.

All the displacements can be related to the parameter θ that represents the rotation of the external column. The relationship between the displacements and the rotation θ can be derived from Figure 5 where are reported the positions of the hinges, the vertical displacements related to the vault at the first level and the horizontal displacements of the external column.

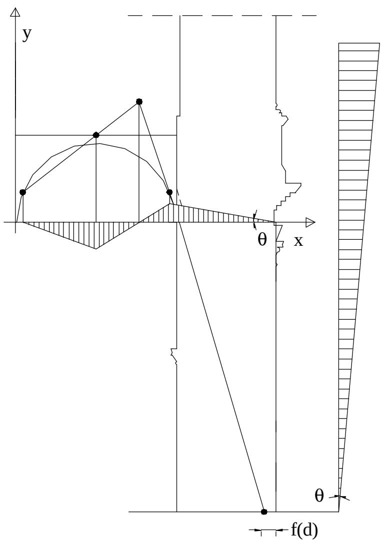


Figure 5: Rotation centres and displacement field related to the first level internal vault.

In the figure f(d) is the distance of the hinge developed at the bottom of the external wall from its external edge. The distance f(d) is assumed different from zero to account for the finite compression capacity of the masonry; it is evaluated considering the fact that the hinge is positioned at the centroid of the crushing area of the column (Como, 1983).

3.2 The mechanism type b)

This type of mechanism involves all the ambulatory vaults and both the external and the central walls. The external and the central walls rotate outwards and the inside wall is fixed because of the rigid support provided by the radial walls. The hinges arrangement in the internal vaults, which are included between the fixed inside wall and the rotating central wall, is similar to that of the external vaults in the mechanism type a). Thus, three hinges will form at each vault, two at the intrados near the abutments and one at the extrados near the crown.
Concerning the external vaults, the possible mechanism can be defined by considering the central and the external walls behaving as two cantilevers with hinges at their base and connected by rigid struts. Then it is possible to distinguish two different arrangements of hinges which characterize the mechanisms named b_1) and b_2) which will be discussed in the next subsections.

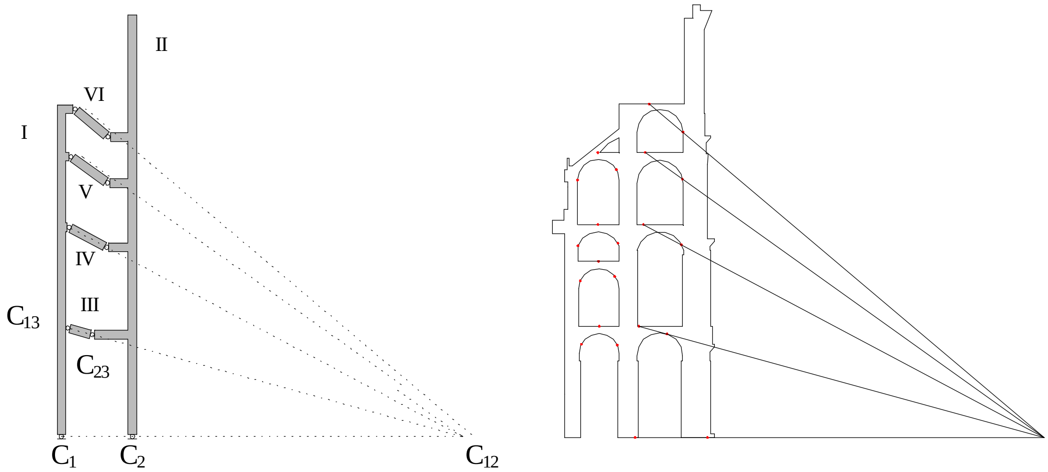
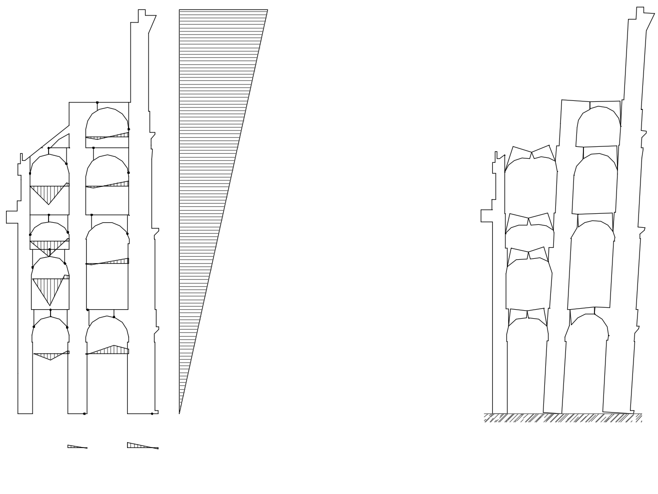


Figure 6: The alignment of the rotation centres to make the structure become a mechanism.

3.2.1 The mechanism type b_1)

A first possible mechanism is defined by considering each external vault behaving as a rigid strut. As it is shown in Figure 7, the directions of the struts are enforced by the alignment of the relative rotation centres C_{1i}, C_{2i} (i=3..6) and C_{12}, in order to make the structure become a mechanism. This condition means that only the positions of the hinges related to one vault are free to be chosen while, for the other vaults, one hinge remains free to be fixed and the other is determined by the intersection between the vault and the line passing through the free hinge and the centre C_{12}.



a) b)

Figure 7: a) vertical and horizontal displacements; b) radial section of the mechanism.

Figure 7 shows a possible mechanism of the structure. All the displacements can be related to the parameter θ which represents the rotation of the central column. The relationship between the displacements and the rotation θ can be derived from Figure 8 where are reported the hinges position and the vertical displacements related to one vault. In the figure f(d) indicates the distance of the hinge at the bottom of the central column from its external edge and $f_1(d_1)$ indicates the distance of the hinge at the bottom of the external column from its external edge.

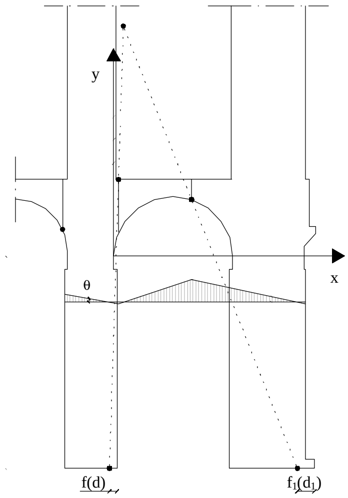
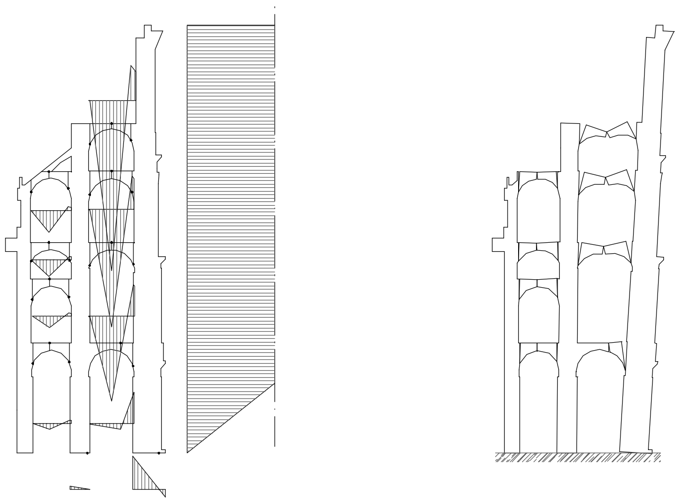


Figure 8: Rotation centres and vertical displacements related to the first level vault.

3.2.2 The mechanism type b_2)

An alternative mechanism of the external vaults is defined by considering only one vault behaving as a rigid strut and the others behaving like the internal vaults with two hinges formed at the intrados of the abutments and one at the extrados of a section close to the crown. In particular it can be shown, although the calculations are not reported herein for brevity, that the mechanism minimising the collapse load is that shown in Figure 9 where the vault at the first level behaves as a rigid strut.



a) b)

Figure 9: a) vertical and horizontal displacements; b) radial section of the mechanism.

Also in this case all the displacements are expressed in function of the rotation θ of the central column. Figure 10 allows to derive the equations relating the displacement components to the parameter θ and to the hinges position.

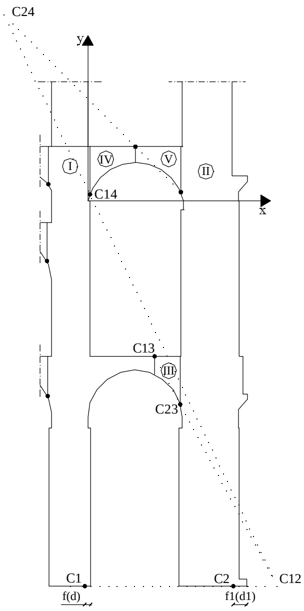


Figure 10: Rotation centres and vertical displacements related to the first level vault.

4 THE COLLAPSE LOAD

4.1 The force system

The force system, acting on the structure, is constituted by the following components:
- the weight W of the walls;
- the dead load p acting on the ambulatory vaults;
- the live load λq acting on the ambulatory vaults;
- the thrust S_c dependent on the dead and live load (Figure 11).

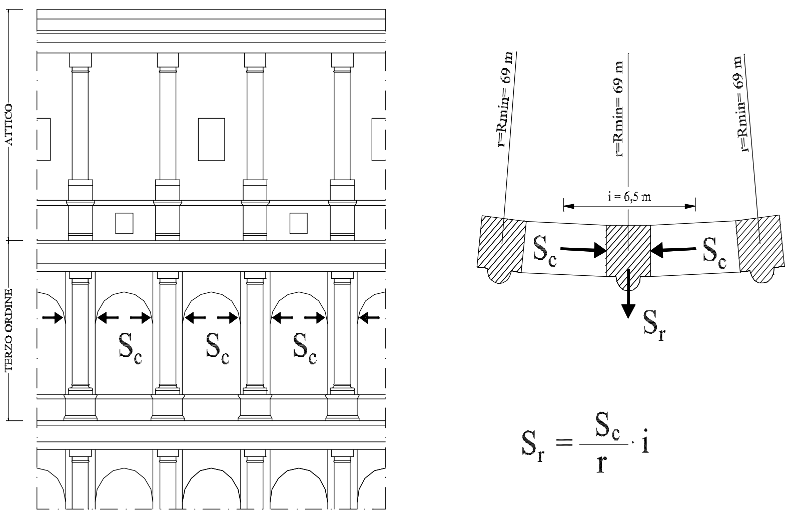


Figure 11: Radial components of the thrusts.

4.2 The evaluation of the collapse load

By applying the kinematic theorem the function λq is given by:

$$\lambda q = \frac{L_{dead}}{L_{live}} \tag{5}$$

The works are evaluated with reference to the structural module shown in Figure 3 and characterised by the radial section reported in Figures 4, 7 or 9.

The work L_{dead} is given by:

$$L_{dead} = L_p + L_W + L_{Sr(p)} \tag{6}$$

where each term is defined as follows.

The first term of (6) is given by the following expression:

$$L_p = \sum_{i=1}^{n} L_{p_i} \tag{7}$$

where L_{p_i} represents the work done by the dead load applied on each ambulatory vault for the relative vertical displacements which are illustrated in Figures 4, 7 or 9 respectively for the mechanism type a), b_1), b_2). The sum is extended to n which represents the number of the vaults involved in the mechanism; thus n=4 in the case of mechanism type a) and n=8 for the mechanisms type b).

The second term of (6) is:

$$L_W = +\sum_{j=1}^{m} L_{W_j} \tag{8}$$

being L_{W_j} the work done by the weight W_j of the central and external walls for the vertical displacements which are illustrated in Figures 4, 7, 9 respectively for the mechanism type a), b_1), b_2). The sum is extended to m which represents the number of the columns in the mechanism; thus m=1 in the case of mechanism type a) where only the external column rotates and m=2 for the mechanisms type b) where both the central and the external columns rotate.

The last term of (6) is the work done by the radial components $S_r(p)$ of the thrusts $S_c(p)$ transmitted by the arches loaded by dead loads (Figure 11):

$$L_{Sr(p)} = \sum_{k=1}^{3} L^c{}_{Sr_k(p)} + \sum_{h=1}^{6} L^e{}_{Sr_h(p)} \tag{9}$$

The first sum is related to the thrusts acting on the central vaults and is equal to zero in the case of the mechanism type a); the second sum refers to the thrusts acting on the external vaults.

The work L_{live} is given by:

$$L_{live} = L_q + L_{Sr(q)} \tag{10}$$

The first term of (10) is given by the following expression:

$$L_q = \sum_{i=1}^{n} L_{q_i} \tag{11}$$

where L_{q_i} represents the work done by the live load applied on each ambulatory vault for the relative vertical displacements which are illustrated in Figures 4, 7 or 9 respectively for the mechanism type a), b_1), b_2).

The last term of (11) is the work done by the radial components $S_r(q)$ of the thrusts $S_c(q)$ transmitted by the arches loaded by the live loads (Figure 11):

$$L_{Sr(q)} = \sum_{k=1}^{3} L^c{}_{Sr_k(q)} + \sum_{h=1}^{6} L^e{}_{Sr_h(q)} \tag{12}$$

The first sum is related to the thrusts acting on the central vaults and is equal to zero in the case of the mechanism type a); the second sum refers to the thrusts acting on the external vaults.

The expression of λq clearly depends on the position of the hinges defining the mechanism. Thus $\lambda q = \lambda q(d, d_1, x_i; i = 1..N)$ where x_i represents the abscissa of the i-hinge and N is the number of the hinges necessary to define the mechanism. The minimum of the function $\lambda q(d, d_1, x_i; i = 1..N)$, evaluated with the constraint expressing the equilibrium of the wall in the vertical direction at a fixed level of the masonry compression strength, provides the collapse load $\lambda_c q$ related to the set of the analysed mechanisms.

With reference to the vertical loading condition analysed in this work, the collapse load $\lambda_c q$, given

by (5) and related to the set of mechanisms type a) and type b), is:
mechanism type a): $\lambda_c q = 11.47$ kN/m^2;
mechanism type b_1): $\lambda_c q = 29.45$ kN/m^2;
mechanism type b_2): $\lambda_c q = 16.56$ kN/m^2.
It is obvious that if the live load acts only on one vault the collapse load becomes greater.

5 CONCLUSIONS

In this paper we analysed the safety condition of the structure of Colosseum under the vertical loading condition. The analyses neglected the soil deformability and the soil structure interaction effects. The study was based on the limit analysis approach developed for masonry structures and adopted a rigid no-tension constitutive model with no sliding. The paper discussed the results of a preliminary investigation referring to the original configuration of the construction. In particular the analysis was performed with reference to some simplified two-dimensional configurations in order to have a better understanding of the response of the principal parts of the whole monument. To this purpose the set of the tree columns defining the radial section of the monument's elliptical plan connected by the ambulatory vaults were analysed and the maximum uniform live load acting on the vaults covering the inner and outer ambulatories were evaluated. The analyses showed a high resistance of the structure under the analysed vertical loading condition.

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