Validated structural analysis of Gothic vaulted systems

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ABSTRACT: A validated analytical procedure for vaulted unreinforced masonry structures is proposed in this study. The structural analyses necessary in this research are carried out by the finite element analysis method (FEM). The ribs of a gothic rib vault are modeled using linearly elastic solid elements, while the vault webs are modeled using linearly elastic shell elements. The analytical models are then validated using in situ nondestructive vibration experiments, by the comparison of analytical and experimental mode shapes and natural frequencies of the vaulting system. Appropriate boundary conditions are determined by the verification of mode shapes, and material properties are verified by comparison of natural frequencies. National Cathedral in Washington, DC, U.S.A. is studied as a controlled case as it has known material properties and construction history. These analysis and validation methods studied on NC will be then applied to European medieval structures in the continuation of this research.

1 INTRODUCTION

Gothic architecture initiated in France in the 12th century and spread through Europe for four centuries. Within the last few hundred years, other structures around the world such as the National Cathedral (NC) in Washington, DC were also constructed in Gothic style. The main component in these unreinforced masonry structures is the complex assembly of vaulted roof systems, for which, there are not well-established and dependable analysis techniques. The available methods are rigid-plastic analyses, elastic analyses, two-dimensional graphical analyses and a current trend of computer analyses mostly based on the finite element method. However, none of these methods has been validated by experiments conducted on the real structures (Boothby, 2001).

A validated analytical procedure for vaulted systems is evident is proposed in this study. The structural analyses necessary in this research are carried out using three-dimensional finite element analysis method (FEM). The analytical models are then validated using in situ nondestructive experiments.

For the case of a Gothic vaulted roof structure, besides the complexity of the structural behavior, material properties and support conditions are unknown. Static load testing or sample extraction, which would reveal information about these parameters are not possible on these historically important structures. Therefore, nondestructive testing is called upon for the validation of FEM. In this study, in situ vibration experiments and experimental modal analyses are used for the updating and validation of the model.

The proposed methods are first studied on a structure with better-known properties and history, the National Cathedral (NC). Therefore, the ribbed vaults in the choir of NC are modeled, the analytical models are experimentally validated, and the results of this study are presented in this paper. The findings, results and improvement suggestions from this study will be applied to similar studies on medieval European structures in the future.

1.1 Washington National Cathedral

The construction of the NC, the sixth largest cathedral in the world, continued for more than 83 years (September 29, 1907 to September 29, 1990). Frederick Bodley, England’s leading Anglican Church architect, was the first head architect with Henry Vaughan as the supervising architect. After Bodley and Vaughan had passed away following World War I, American architect Philip Hubert Frohman became the principal architect. The cathedral was built in the Decorated Gothic style, a style in English architecture that lasted from 1270 to 1380. The cathedral was engineered and constructed like the medieval churches, using only quarried stone with no steel reinforcement in any part of the building and no mass-produced elements. Even though the exact dates are unknown, the choir vaults, which will be analyzed in this study, were finished around the 1930’s, when the transepts were opened for public use. The even-level-crown,
2.1 Finite element model material properties

The models are created with the assumptions of linear behavior and homogeneous material. The initial assumption for linear behavior is proved to be appropriate after the review of dynamic experiment results, because the impacts from different magnitudes of impact (heel drops vs. hammer impact) generate similar behavior on the structure, as will be presented later in this paper. The homogeneous material assumption, which considers effective material properties for the masonry unit and mortar assembly rather than separate entities for units and masonry joints, is adopted for simplification of the finite element analysis procedure.

The material properties are gathered from various sources such as literature (www.langstone.com), laboratory experiments on an Indiana limestone sample, and calculations to account for the mortar unit assembly assumption. Depending on the stiffness of the mortar and the thickness of the mortar joint, the effective stiffness of the assembly varies. Table 1 summarizes the possible Modulus of Elasticity (E-value) ranges for masonry assembly with type O mortar and Indiana limestone. The mortar E-values are from McNary & Abrams (1985) and E value for the Indiana limestone is from laboratory testing. The mortar joint thicknesses vary between 9.5 mm and 6.4 mm (3/8" - 1/4").

According to the information given in Table 1, the modulus of elasticity for the masonry-mortar assembly for the finite element model ranges between 8 GPa to 21 GPa, depending on the condition and thickness of the mortar joints. When the model is updated by the experimental results, the optimum value for the masonry assembly stiffness is found to be 12 GPa. This is partially due to the fact that the masonry in the vaults
Table 2. Final material properties for NC finite element model.

<table>
<thead>
<tr>
<th>Material</th>
<th>Modulus of elasticity (GPa)</th>
<th>Density (kg/m³)</th>
<th>Poisson's ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Masonry assembly</td>
<td>12</td>
<td>2100</td>
<td>0.2</td>
</tr>
<tr>
<td>Concrete surcharge</td>
<td>24</td>
<td>2100</td>
<td>0.15</td>
</tr>
</tbody>
</table>

Figure 3. Final boundary conditions as updated by the experimental findings.

The other material properties required are the density and the Poisson's ratio of the masonry assembly, and the material properties for the surcharge, which is concrete in the case of NC. Table 2 presents the experimentally validated final values for these parameters.

2.2 Finite element model boundary conditions

The boundary conditions are applied based on observations and are then updated by experimental results to match the experimentally observed mode shape. The final boundary conditions, as illustrated in Figure 3, are as follows: The sides of the arches are restrained in the Y direction to account for the adjacent vaults. The top portion of the side arches is restrained in the vertical direction to account for the vertical-movement-limiting effects of the adjacent arches. Parts of the exterior wall are restrained in the X direction to account for the wall buttresses, flying buttresses and connection of the roof trusses to the exterior wall.

3 EXPERIMENTAL PROCEDURES AND RESULTS

In this research, the experimental modal analysis techniques are used to validate the finite element models of the complex vaulted systems. This is an application of dynamic experiments that the design of experiments constitutes an important part of the procedure. Therefore, experience from similar experiments performed on other types of structures is called upon. Hanagan et al. (2003) investigate the variables that affect experimental modal analysis of floor systems, some of which are discussed in the following sections and are used in this study to determine the experimental procedures.

3.1 Experimental equipment and setup

In this study, three different excitation methods are used: heel drops, shaker excitation and hammer impacts. The heel drop is a man-made impact created by raising up on one's toes and dropping the heels onto the floor. The heel drop impact force is measured using a force plate constructed by attaching four shear beam load cells to the corners of a thick aluminum plate (Hanagan, et al., 2003). The force plate used for heel drops in this study, has a calibration coefficient of 3916 N/volt. The instrumented hammer is another convenient tool to excite a floor system because of its portability. The instrumented hammer used is PCB Piezetronics ICP® Impulse Hammer Model 086C20. The hammer has options for creating the impact, including a selection of tips with varying hardness. Since the hammer is instrumented, the force plate is not required and the hammer can be connected directly to the data acquisition system. However, in this set of experiments another custom designed force plate, with a calibration coefficient of 801 N/volt is used to measure the force created by the hammer.

The force plate, which is used for the heel drops, is also used for the shaker. Due to the heavy weight and low coherence caused by some technical problems with the available shaker at hand, the shaker is not used in these experiments extensively. However, as better data is available from shaker excitation for one of the setups (Row4), the data from that experiment is used for determining the mode shape and is presented in this paper as evidence for the consistency of the results regardless of the excitation type and thus the linearity of the structural behavior.

Seven PCB Piezetronics model 393A03 seismic accelerometers, with a sensitivity of 1 V/g are used as vibration transducers and were placed in various arrays. The experiments on the NC vaults cover
three bays of the choir with the driving point on the crown of the third bay counted from the crossing. An experimental setup plan and the picture from the experimental site are shown in Figure 4. A schematic representation of the experimental setup is also provided on the finite element model in Figure 2.

Other variables that affect experimental modal analysis of floor systems as investigated by Hanagan et al. (2003), and are pertinent to this study are briefly discussed below.

A SIGLAB dynamics signal analyzer is used for collecting and processing the data. This dynamic signal analyzer includes anti aliasing filters to eliminate the high frequency signals and allows for real-time post-processing and adjustments.

For efficient data processing, considerations such as the selection of the record length, bandwidth, averaging and windowing are used to achieve a good frequency resolution, to adequately capture the data range of interest, to distinguish the data from the ambient noise and to avoid leakage, respectively. All of these are set-up using the SIGLAB software graphical interface.

3.2 Experimental data analysis and results
The parameters that can be extracted through vibration experiments and would be useful for comparison with analytical models are the mode shape, the frequency for the observed mode and an approximation for the system stiffness. Consequently, two types of experimental outputs are of concern: frequency response functions (FRF) and Coherence (COH). FRF is the complex function of the ratio between a harmonic displacement response and the harmonic force in the frequency domain (Raebel, 2000). Using a force plate or the load cell within the hammer, the force input is measured and thus the FRF is plotted in terms of the normalized acceleration amplitude. Since it is a complex function, an FRF can be expressed by the real and imaginary parts, or in polar coordinates as the phase angle and magnitude. All of these possible presentations of an FRF reveal different information about the modal behavior of the structure.

The magnitude and phase plots of the FRF reveal the resonant peaks of the response and the corresponding 90° phase shift at the natural frequency of the observed

Figure 4. The experimental setup shown on the partial floor plan and the photo of the experiment site above the vaults.

Figure 5. FRF magnitude and coherence plot for heel drops for the NC experiment setup col4.

Figure 6. FRF phase plot for the NC experiment setup col4.
mode of the system. Figures 5 and 6 present these plots for the heel drop tests in the NC, for the sensor located immediately adjacent to the driving point for the setup shown in Figure 4. As can be seen from both plots, there is an apparent mode with a natural frequency of 36 Hz.

Since the force input is normalized and plotted for every frequency, FRF allows for averaging and comparing data from several excitations even though the applied force intensity varies. The coherence function measures how well the output is linearly related to the input for every frequency, and is rated on a 0 to 1 scale. Coherence measurements approaching unity indicate a linear trend between output and input, and in most cases represent high confidence in the measured data (Hanagan et al., 2003; Raebel, 2000). The coherence for the heel drop measurements for the sensor next to the driving point is presented in Figure 5 and it suggests good quality data except the 0–5 Hz low frequency range, where the ambient noise interferes with the generated vibrations.

Figure 7 shows the FRF from a hammer impact at the same driving point. As can be seen, the low coherence for hammer impact covers a larger range of lower frequencies (0–25 Hz). This is mainly because the hammer impact is a low amplitude excitation, that can generate lower frequency vibrations and therefore the data is more susceptible to the ambient noise. On the other hand, the FRF plots for these two different excitation levels are very similar, which supports the assumption of linear behavior for this structural system.

The deflected shape of the structure from the experimentally observed mode shape is gathered from the imaginary plots, as this representation of the FRF also includes the sense of the acceleration for each sensor (Figs 8 & 9). By noting the positive and negative motion of each sensor, the deformed shape for the testing scheme illustrated in Figures 1 and 4 is plotted in Figure 10.

By comparing this experimentally observed deformed shape with the analytically established mode shapes, it is possible to validate the boundary condition assumptions, because the mode shape is determined mainly by boundary conditions. Other parameters such as the material properties (Masonry and surcharge stiffness, density and Poisson’s ratio) have no effect on the mode shape, although they affect the natural frequency of this mode shape.

An additional experimental result of interest is the FRF plotted in displacement units, because the displacement at zero frequency gives the inverse of the system stiffness, i.e. the system flexibility. Since the sensors measure acceleration, the data collected must be converted to displacement units.

![Figure 7](image7.png)  
**Figure 7.** FRF magnitude and coherence plot for hammer impact for the NC experiment setup col4.

![Figure 8](image8.png)  
**Figure 8.** Imaginary plot for heel drop excitation for all sensors in the NC experiment setup col4.

![Figure 9](image9.png)  
**Figure 9.** Imaginary plot for shaker excitation for all sensors in the NC experiment setup row4.

![Figure 10](image10.png)  
**Figure 10.** Deformed shape of the top beams detected by the grid of accelerometers at 36 Hz.
using the relationship between the acceleration and displacement as shown by Equation 1.

\[ \ddot{X} = -\omega^2 X \quad \text{or} \quad X = \frac{\ddot{X}}{-\omega^2} \]  

Where, \( \ddot{X} \) is the acceleration  
\( X \) is the displacement  
\( \omega \) is the circular natural frequency  

The data collected by the accelerometer next to the driving point during heel drops is converted to displacement units by Equation 1, and plotted as shown in Figure 11.

If the acceleration at zero frequency were actually zero, then the equation would result in a singularity and would be solved for by an indefinite integral solution. Thus, the conversion between the acceleration and displacement units in theory is not trivial but it is possible. However, in practice, this conversion generates some problems. The ambient noise results in very small vibration amplitudes at the lower frequencies, which is also apparent from the low coherence between 0-5 Hz, as shown in Figure 5. Therefore, the displacement plot is not reliable at these lower frequencies. But if a displacement FRF for a simple system is inspected, it is observed that there is a plateau starting at about the root of the resonant peak of a natural frequency and ending at 0 Hz. Therefore, the value at 25 Hz can be extrapolated in the plot in Figure 10 to 0 Hz and hence result in an approximate value for the system stiffness of 5E-9. Although approximate, this system stiffness value can then be compared to analytically established values to see if they are in the same order of magnitude.

4 MODEL VALIDATION

The finite element model is analyzed by linear elastic, undamped modal analysis for a comparison with the experimental results. The modal analysis equation used by the finite element analysis software ANSYS is given by Equation 2 (ANSYS, 2003).

\[ [K] \{\varphi_i\} = \omega_i^2 [M] \{\varphi_i\} \]  

Where, \([K]\) is the stiffness matrix  
\( \{\varphi_i\} \) is the mode shape vector (eigenvalue) of mode \( i \)  
\( \omega_i \) is the natural circular frequency of mode \( i \)  
\( \omega_i^2 \) is the eigenvalue  
\([M]\) is the mass matrix  

When such an analysis is performed on the three-vault model of NC, the 5th mode of vibration, which is the lowest mode with a substantial vertical movement at the sensor points, presents a similar deformed shape as the experimentally observed one, at 35.5 Hz. These results are for the previously described final material properties and boundary conditions, and present a very good agreement with the experimental results. Hence the model is validated with the in situ experiments. The results for this mode shape are shown in Figure 12 on the full model and Figure 13 shows the deformed shape for the top ribs only.

If Figure 13 is compared to Figure 10, it is seen that the analytically determined mode shape for the top ribs are in very good agreement with the experimentally observed deformed shape.

In order to determine the system stiffness analytically, a static unit load is applied to the point, which is the driving point during the experiments and the displacement at this point is sought. Figure 14 presents the graphical results for this static analysis.

The system flexibility (inverse of system stiffness) determined by this analytical model is 4E-09 m/N while the experimentally gathered approximate value is 5E-09 m/N. The two are in the same order of magnitude. The error is due to the low coherence of
5 SUMMARY AND DISCUSSIONS

In this study, the experimentally observed mode shape and the natural frequency of this mode are compared to the analytical model results for the validation of analytical models. As the applied boundary conditions result in a mode shape that is in very good agreement with the experimentally observed mode, the next step is to determine the most accurate material properties within the appropriate range, which is gathered through a combination of lab experiments and literature. Effective stiffness values are calculated to account for the combined properties of the Indiana lime stone and type O mortar. With these material properties, the analytical natural frequency is in very close agreement with the experimentally recorded natural frequency. The analytical system stiffness is also compared to an approximate experimental value, which are in the same order of magnitude. Although there is 25% error in this comparison, this error is mostly due to the approximately calculated value of the displacement at zero frequency. As a result of all three types of comparisons and updating, the analytical model is successfully validated by the nondestructive vibration experiments.

Besides proposing a validation method for the complex three-dimensional finite element models of the stone vaulted systems, this study reveals several useful facts about the dynamic testing of such structures. For instance, it is observed that the heel drops give more coherent results than the impulse hammer impacts. However, as heel drops are not possible on most vaulted systems, an instrumented hammer, which presents the advantages of convenience and portability, is recommended, since it gives comparable results for the mode shape and natural frequency.

Another finding through various experimental trials in this study is that, the structure behaves linearly under dynamic loading and thus linear analysis on the structure is appropriate.

The work presented in this study is an excerpt from an ongoing research project, but it reveals some very useful conclusions for the future work of the study, which explores the extrapolation of the proposed methods to European medieval structures. For these cases where the material properties are also unknown, the vibration experiments on the structure itself and the knowledge from similar structures with calibrated models, such as established here for NC, can be used for model validation.

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