Comparison of the masonry structures analysis using the co-rotational formulation and a simplified proposal

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ABSTRACT: The purpose of the present article is to compare the analysis of masonry structures using the co-rotational formulation and a simplified proposal to calculate the second order effects. In the simplified proposal the non-linear displacement are obtained as:

\[ \{d_{NL}\} = \{d\} + \{a_i\}/\{P/\{P - 1\}\}\{\phi_i\} \]

where \(\{d\}\) are the linear displacement, \([K_d]\) are the buckling mode associated with the smaller buckling load \(P_1\) \([K_G] + P_1/P \cdot [K_d]\) \(\{\phi_i\} = [0]\). The coefficients \(\{a_i\}\) are given by:

\[ \{a_i\} = ([\phi_i]^T\{F_{ext}\})/([\phi_i]^T[K_d][\phi_i]) \]

these can be easily obtained using the orthogonality properties of the buckling modes respect to the linear stiffness matrix \([K_L]\) and the geometric matrix \([K_G]\).

From the various examples presented it can be concluded that the simplified proposal provides very accurate results. Furthermore the time required to solve the analysis as well as the number of parameters used is smaller, this means that it will be easier to control the physical meaning of the results.

1 INTRODUCTION

The objective of the present article is to compare the analysis of masonry structures using the co-rotational formulation and a proposal that calculates the second order effects in a simplified form.

The advantages of this simplified proposal are:

1. Provides very accurate results so this method can be used to perform a nonlinear geometric analysis of the structure.
2. Furthermore the time required to solve the analysis is smaller.
3. As well as the number of parameters used is smaller, so it will be easier to control the physical meaning of the results.

Many practical applications can be performed, for example 4.1.2 could be used to analyse the second order and imperfections effects on a masonry chimney.

Three are the kinematics descriptions currently used to describe the kinematics equations and the equilibrium equations these are: The “Total Lagrangian” (TL), the “Update Lagrangian”(UL) and the “Co-rotational” (CR).

In first place a brief resume of the co-rotational formulation is presented. In second place the proposed method is described. In third place some examples are used to validate the simplified proposal.

The Co-rotational formulation has its roots in the old idea of splitting the total movement into a rigid solid movement and a purely deformational. Two reference configurations are used: The unloaded and the Co-rotational, this is obtained through a rigid solid movement from the unloaded configuration. The deformation of the element is measured from the Co-rotational configuration and is admitted that the movements referred to such configuration are very small, so it is allowed to apply the linear kinematics. When this hypothesis is not fulfilled, in some element, these must be divided into more. This formulation presents the advantage that uncouples the material nonlinearities from the geometric nonlinearities. The material nonlinearities are taken into account by considering the behaviour with respect to the co-rotational configuration and the geometric nonlinearities are incorporated into the analysis by considering the rigid solid movement of the element axes.

2 CO-ROTATIONAL FORMULATION

The Co-rotational approach has roots on a very old idea: the separation of rigid body and purely
deformational motions in solid and continuum mechanics. It originally arose in theories of small deformations coupled with large rigid motions.

Given the global displacements: (U₁, V₁, W₁, ϕ₁, q₁, Φ₁) and (U₂, V₂, W₂, ϕ₂, q₂, Φ₂); (U, V, W) are movements about axes x, y and z; (ϕ₁, q₁, Φ₁) are the total rotations about x, y and z, Φ is the warping where the rotation matrix is given by Rodrigues (1840):

\[ R_T = e^{[\phi]} = [I] + \frac{\sin(\phi)}{\phi} [W] + \frac{1}{2} \left( \frac{\sin(\phi/2)^2}{\phi/2} \right) \cdot [W]^T \]

\[ [T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ [W] = \begin{bmatrix} \phi_x & \phi_y & \phi_z \\ \phi_y & \phi_z & \phi_x \\ \phi_z & \phi_x & \phi_y \end{bmatrix} \]

The rigid solid rotation \([RSR]\) rotates the unloaded local axes \([T_A]\) of the element into the new configuration \([T_B]\): \[ [T_B] = [RSR] \cdot [T_A] \]

The deformational displacements \(\{\theta_1, \theta_2, \Phi_1, \phi_1, \theta_3, \phi_3, \Phi_2\}\) can be obtained extracting the rigid solid from the total: \[ [R_D] = [R_T] \cdot [RSR]^{-1} \cdot [R_T] \cdot [RSR]^T \]

As the deformational rotations are small these can be expressed as:

\[ [R_D] = [I] + [W] = \begin{bmatrix} 1 & -\theta_x & \theta_y \\ \theta_x & 1 & -\theta_y \\ -\theta_y & \theta_x & 1 \end{bmatrix} \]

\(u_2 = L_f - L_0\); where \(L_f\) is the final length and \(L_0\) is the initial length. \((\theta_x, \theta_y, \theta_z)\) are the deformational rotations about x, y and z.

3 DESCRIPTION OF THE PROPOSED METHOD

The method presented in this paper is based on:

1. A linear analysis of the structure used to obtain the primary internal forces and moments.
2. An initial stability analysis used to obtain the secondary internal forces and moments.
3. The orthogonality properties of the buckling modes to simplify the analysis.
\[(v_i, w_i, \theta_{xia})\] is the particular solution due to imperfections.

If the imperfection is given by the first bucking mode the solution associated with the imperfection is:

\[v_{cia} = \frac{v_i}{\mu_{1MN} - 1}; w_{cia} = \frac{w_i}{\mu_{1MN} - 1}; \theta_{xia} = \frac{\theta_{x1}}{\mu_{1MN} - 1}\]

The consistent internal forces are:

Axial:

\[N = EA \frac{du}{dx}\]

Shear:

\[Q_y = -Ez \frac{d^3 v}{dx^3} - My \frac{d(\theta_y + \theta_z)}{dx} - p \frac{d(v + v_i)}{dx}\]
\[Q_z = -Ely \frac{d^3 w}{dx^3} - p \frac{d(w + w_i)}{dx}\]

Torsional moment:

\[M_x = \frac{d^3 \theta_y}{dx^3} + Glt \frac{d\theta_z}{dx} - My \frac{d(\theta_y + \theta_z)}{dx} - Pr_t \frac{d(\theta_y + \theta_z)}{dx}\]

Bending moment:

\[M_y = -Ely \frac{d^2 w}{dx^2}\]
\[M_z = Elz \frac{d^2 v}{dx^2}\]

Bimoment:

\[B_i = ELa \frac{d^3 \theta_y}{dx^3}\]

The relationship between forces and displacements, stiffness matrix is given by:

Extension:

\[f_{x1} = \begin{bmatrix} \frac{E - A}{L} & -\frac{E - A}{L} \\ \frac{-E - A}{L} & \frac{E - A}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} f_{x1} \\ f_{x2} \end{bmatrix}\]

Strong bending:

\[f_{x1} = \frac{k^2 - Ely}{2 - 6e - kLs} \begin{bmatrix} s^k & c^k \\ c^{k-1} & s^{k-1} \end{bmatrix} + \begin{bmatrix} s^k & c^k \\ c^{k-1} & s^{k-1} \end{bmatrix} \begin{bmatrix} w_i \\ \theta_{x1} \end{bmatrix} + \begin{bmatrix} f_{x1} \\ f_{x2} \end{bmatrix}\]

Weak bending coupled with torsion:

\[f_{y1} = \begin{bmatrix} v_1 \\ \theta_{x1} \end{bmatrix} + \begin{bmatrix} f_{y1} \\ M_{elas} \end{bmatrix} = [A][B]^T = 0\]

\[B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta & 0 & \psi & 0 & 0 & 0 & 0 \\ 0 & \mu & \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu & \eta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta & \gamma & 0 & 0 & 0 & 0 \\ 0 & \eta & \mu & \beta & \eta & \gamma & 0 & 0 \\ 0 & \mu & \eta & \gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta & \gamma & \eta & \gamma & 0 & 0 \end{bmatrix}\]

3.2 Simplified method to obtain the second order effects

The orthogonality properties of the buckling modes \((\phi_i)\) are for a general section:

In a column:

\[\int P \cdot (\phi_i \cdot \phi_j + \phi_i \cdot \phi_j + \gamma^2 \cdot \phi_i \cdot \phi_j - 2 \gamma c \cdot \phi_i \cdot \phi_j) \, dx = \langle \phi_i \rangle \langle \phi_j \rangle [K_{GM}] [\phi_i] = 0\]

\[\int [E - I_x \cdot \phi_{i1} \cdot \phi_{j1} + G - I \cdot \phi_{i2} \cdot \phi_{j2} + E - I \cdot \phi_{i1} \cdot \phi_{j2}] \, dx = \langle \phi_i \rangle \langle [K_1] \phi_i \rangle = 0\]

\[ro^2 = \frac{Ey + Ez}{A} + \gamma c + \gamma c\]

In a beam:

\[\int [2 - 2M_y (\beta \gamma \phi_{i1} \cdot \phi_{j1} + \phi_{i1} \cdot \phi_{j1}) - 2 - 2y \gamma \phi_{i1} \cdot \phi_{j1}] \, dx = \langle \phi_i \rangle \langle [K_{GM}] \phi_i \rangle = 0\]

\[\int \left[ E - I_x \cdot \phi_{i1} \cdot \phi_{j1} + G - I \cdot \phi_{i2} \cdot \phi_{j2} + E - I \cdot \phi_{i1} \cdot \phi_{j2} \right] \, dx = \langle \phi_i \rangle \langle [K_1] \phi_i \rangle = 0\]

\[\beta y = \frac{1}{2} \frac{Ey}{4} \int (y^2 + z^2) \, dx - \gamma c\]

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The nonlinear displacements can be expressed using these properties as:

\[ \{d_{NL}\} = \{d\} + \sum_{j} \frac{a_j}{(\mu_j - 1)} \{F_j\} + \frac{F_{squeez}}{(\mu_s - 1)} \{\phi_1\} \approx \{d\} + \frac{a_1 + F_{squeez}}{(\mu_1 - 1)} \{\phi_1\} \]

\[ \{a_j\} = \{\phi_j\} \{K_j\} \{F_j\} \]

In a structure with primary bending moments and axial forces that may induce buckling, the nonlinear displacements can be obtained as:

\[ \{d_{NL}\} = \{d\} + \frac{a_{MN}}{(\mu_{MN} - 1)} \{\phi_{MN}\} + \frac{F_{squeez}}{(\mu_{MN} - 1)} \{\phi_{MN}\} \]

\[ \{a_{MN}\} = \left\{ \frac{E I_x}{E I_x + E I_y} \phi_{MN} + E I_x \phi_{MN} + G I_z \theta_y \phi_{MN} + E I_y \phi_{MN} \right\} \right\} \int \left[ \left( \frac{E I_x}{E I_x + E I_y} \phi_{MN} \right)^2 + \left( E I_x \phi_{MN} \right)^2 + G I_z \phi_{MN} \right] dx \]

\[ = \{\phi_{MN}\} \{F_{MN}\} \]

\[ \{\phi_{MN}\} = \{K_{MN}\} \{a_{MN}\} \]

4 EXAMPLES

Some examples are presented to validate the simplified proposal.

4.1 Example 1. A cantilever column is studied under several loadings:

4.1.1 First case: the geometry and loading is shown in figure 1

The nonlinear displacements with the proposal are:

\[ a_1 = \left[ \frac{E I_x}{E I_x + E I_y} \phi_1''(x) \phi_1''(x) \right] \frac{dx}{dx} = \frac{16 M L^2}{\pi^2 E I_z} \]

\[ v_{NL} \approx \frac{M x^2}{2 E I_z} + \frac{16 M L^2}{\pi^2 E I_z (\mu_1 - 1)} \left( 1 - \cos \left( \frac{\pi x}{2 L} \right) \right) \]

Figure 2. Rate between secondary and primary internal bending moments.

The nonlinear "analytical" displacements are:

\[ v_{NL} = \frac{M}{P \cos(k L)} \left( 1 - \cos(k x) \right) \]

4.1.2 Second case: the geometry and loading is shown in figure 3

The nonlinear displacements with the proposal are:

\[ a_1 = \left[ \frac{E I_x}{E I_x + E I_y} \phi_1''(x) \phi_1''(x) \right] \frac{dx}{dx} = \frac{32 H L^3}{\pi^4 E I_z} \]

\[ v_{NL} = \frac{H x^2}{6 E I_z} + \frac{32 H L^3}{\pi^4 E I_z} \left( 1 - \cos \left( \frac{\pi x}{2 L} \right) \right) \]

The nonlinear "analytical" displacements are:

\[ v_{NL} = \frac{H}{P} \cos(k x) \left( L - \frac{c k L - s}{c k} + \frac{\sin (k x)}{k} + L - x + \frac{s - c k L}{c k} \right) \]

Figure 1. Loading and geometric definition.

Figure 3. Loading and geometric definition.
4.2 Example 2. The simply supported beam shown in figure 5 is studied.

The nonlinear displacements with the proposal are:

\[ a_i = \left( \frac{E' I_z v'' + G' I_t \theta_x + E' I_a \theta_x'' + G' I_t \theta_x'}{\left( E' I_z (\phi_{i1}'' + G' I_t \phi_{i1}) + E' I_a (\phi_{i1}''' + G' I_t \phi_{i1}') \right)^2} \right) \int_{\theta_{i1}}^{\theta_{i2}} dx \]

\[ = \frac{2L}{\pi^2 E I z} = \frac{L^2 M_{zo}}{2 \pi^2 E I z} \]

\[ \begin{bmatrix} v \\ \theta_x \end{bmatrix} = \frac{M_{zo}}{P_z} \sin \left( \frac{\pi x}{L} \right) \left[ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2(\mu_i - 1)} \left( \frac{1}{P_z} \right) \right] \]

The nonlinear "analytical" displacements are:

\[ \begin{bmatrix} v \\ \theta_x \end{bmatrix} = \frac{M_{zo} \sin \left( \frac{\pi x}{L} \right)}{M_{cr}^2 - M_y^2} \left( \begin{bmatrix} G l t + \pi^2 E I a \\ L^2 \end{bmatrix} \right) \]

\[ M_{cr} = \frac{\pi^2 E I z}{L^2} \left( \frac{I_a + L^2 G I t}{\pi^2 E I z} \right) \]

\[ P_z = \frac{\pi^2 E I z}{L^2} \]

4.3 Example 3. The Frame shown in the figure 7 is studied.

The nonlinear displacements with the proposal are:

\[ a_i = \left\{ \begin{array}{l} \{ P_{ext} \} \\ \{ \phi_i \} \end{array} - \left[ \begin{array}{l} \{ P \} \\ \{ L \} \end{array} \right] \right\} = \frac{H L}{L} \left( \begin{array}{l} \frac{13}{15} + \sqrt{464} \\ 15 \end{array} \right) \]

\[ \Delta = H L \left( \left[ \begin{array}{l} 2200 - 646 - 51680 \end{array} \right] \left[ \begin{array}{l} 646 + 11 \end{array} \right] \left( \begin{array}{l} 4 \end{array} \right) \right) \]

\[ \theta_1 = \theta_2 = \frac{H L \left( \begin{array}{l} 2440 - 646 - 103360 \end{array} \right) \left( \begin{array}{l} 4 \end{array} \right) \left( \begin{array}{l} 4 \end{array} \right) \right) \]

Figure 4. Rate between secondary and primary internal bending moments.

Figure 5. Loading and geometric definition.

Figure 6. Amplification factor of the weak bending moments is shown ("analytical" (red) - proposed (green)).

Figure 7. Loading geometric definition.
The nonlinear “analytical” displacements are:

\[
\{\Delta \theta, \theta_\ell\} = \frac{-H L^2}{2[P L^2 (s + s) - E L^2 (s + s^2 - s^2)]} \left\{\begin{array}{c}
(s + 6)L - s(s + c) \\
(s + 6)L - s(s + c)
\end{array}\right. 
\]

5 CONCLUSIONS

From the examples presented can be concluded that the simplified proposal provides very accurate results so this method can be used to perform a nonlinear geometric analysis of the structure. Furthermore the time required to solve the analysis as well as the number of parameters used is smaller, this means that it will be easier to control the physical meaning of the results.

REFERENCES


