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On the Stability of Stone Arches

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ABSTRACT: In the recent years a growing interest to historical structures has motivated the development of new numerical procedures able to investigate the mechanical behaviour and to predict mechanism of failure with rigorous theoretical models, in order to estimate the safety and the remaining life of the structures [1-4]. The present work is direct to analyse the mechanism of collapse in monumental arches made of stone in contact without interposition of mortar [5,6]. The structure is supposed to be in an initial rest configuration, corresponding to the *just built* shape, where it lies in a condition of stability under its own weight.

1 INTRODUCTION

The failure consists of a loss of stability caused by a geometrical perturbation of the reference configuration outside the control of its cause. The arch structure is identified by its geometrical characteristics: shape, number of voussoirs, position of hinges, thickness, span; the exact position of all the hinges can be determined, by analytical way too, knowing the constitutive law of the axis curve of the arch and their geometrical position, according to a compatible mechanism. The 3D model is generated from a rectangle moving along a parabolic curve that represents the axis curve of the arch.

The research of this limit configuration is followed by an energetic approach: in the initial stabile configuration C_0 the potential energy of the system assumes the same value of the work carried out by a load equivalent to the own weight of the structure. The initial position represents a situation of stability, in fact there is no displacements and so the potential energy too. If the system is subjected to unilateral constrains, a perturbative kinematic parameter φ capable to provide $dP/d\varphi>0$, can represent a measure of degree of stability of the structures. The results are obtained by a mathematical programming procedure, ad hoc formulated, first by investigating the perturbation amplitude in assigned position of the hinges cases and then assuming the position as a variable defined by some parameters, to obtain the mechanism related to the minimum amplitude of the critical perturbation. Finally a dynamic approach is studied using the Lagrange equation and obtaining the expression of angle φ with respect to the time.

2 THE PROPOSED MODEL

This work studies the mechanic behaviour of a system made of rigid blocks in contact without the interposition of mortar and analyze the stability of the structure.

It is assumed that the collapse of the arch depend on the loss of stability of the system and not on the breaking of the material that is supposed infinitely resisting.

In the work is analyzed an arch made of three units (Fig. 1). The model can has only one possible collapse mechanisms (Fig. 2), furthermore it represent an arch and, for $h \to \infty$, a portal.

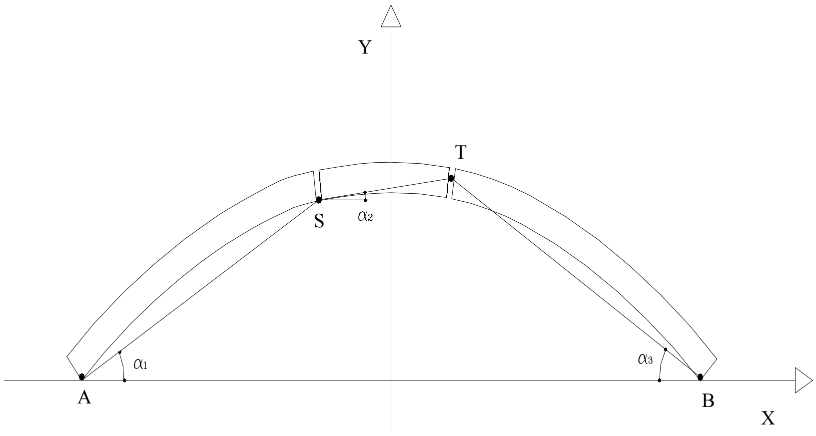

Figure 1 : The proposed model

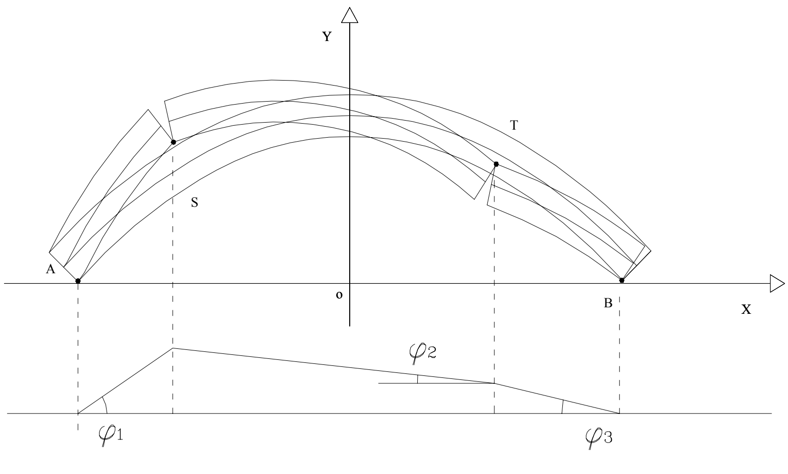

Figure 2 : The collapse mechanism

3 AN ENERGETIC APPROACH

The research of limit configuration can be obtained by an energetic approach: it defined the potential energy of the system under study, by means of a perturbation of the lagrangian parameter that define the mechanism. The potential energy has a stationary point in $\varphi = \varphi^*$, this value is the limit between a stable and unstable behaviour of the system: with $\varphi < \varphi^*$, removing the perturbation, the system return to the initial configuration C_0; with $\varphi > \varphi^*$, removing the perturbation, the structure goes away from the equilibrium configuration.

Defining $\alpha_1, \alpha_2, \alpha_3$ the angles represented in figure 1 and a_1, a_2 and a_3 the length of $\overline{AS}$, $\overline{ST}$ and $\overline{TB}$, from the consideration that the position of points A and B doesn't vary with time it is possible to write the following expressions:

$$
\begin{aligned}
z_B - z_A &= a_1 \cos(\alpha_1 + \varphi_1) + a_2 \cos(\alpha_2 + \varphi_2) + a_3 \cos(\alpha_3 + \varphi_3) \\
y_B - y_A &= a_1 \sin(\alpha_1 + \varphi_1) + a_2 \sin(\alpha_2 + \varphi_2) + a_3 \sin(\alpha_3 + \varphi_3)
\end{aligned} \tag{1}
$$

deriving the equation (1) with respect to φ we obtain

$$
\begin{aligned}
a_1 \sin(\alpha_1 + \varphi_1) d\varphi_1 + a_2 \sin(\alpha_2 + \varphi_2) d\varphi_2 + a_3 \sin(\alpha_3 + \varphi_3) d\varphi_3 = 0 \\
a_1 \cos(\alpha_1 + \varphi_1) d\varphi_1 + a_2 \cos(\alpha_2 + \varphi_2) d\varphi_2 + a_3 \cos(\alpha_3 + \varphi_3) d\varphi_3 = 0
\end{aligned} \tag{2}
$$

from which

$$
\begin{aligned}
a_1 \sin(\alpha_1 + \varphi_1) d\varphi_1 &= -a_2 \sin(\alpha_2 + \varphi_2) d\varphi_2 - a_3 \sin(\alpha_3 + \varphi_3) d\varphi_3 \\
a_1 \cos(\alpha_1 + \varphi_1) d\varphi_1 &= -a_2 \cos(\alpha_2 + \varphi_2) d\varphi_2 - a_3 \cos(\alpha_3 + \varphi_3) d\varphi_3
\end{aligned} \tag{3}
$$

The equation (3) represents the relation existing between the parts of the arch. The procedure takes into account an iterative analysis, fixing, at the first step, $\varphi_1 \cong 1$ and using the following approximation

$$
\begin{aligned}
\sin\varphi &\cong \varphi \\
\cos\varphi &\cong 1
\end{aligned} \tag{4}
$$

Finally it is possible to obtain the displacements of a generic point as

$$
\begin{aligned}
v_i &= a_i \sin\alpha_i - a_i \sin(\alpha_i + \varphi_i) \\
w_i &= a_i \cos(\alpha_i + \varphi_i) - a_i \cos\alpha_i
\end{aligned} \tag{5}
$$

Determined the displacements we could determinate the expression of the potential energy P as follows:

$$
\mathrm{P} = -\left(\mathrm{F_y v} + \mathrm{F_z w}\right) \tag{6}
$$

Ones known the function P it is possible to calculate the stationary point and the value of φ^*.

With reference to the case of figure 1, where the system need of one lagrangian parameter, the parametric analysis provides the definition of the smallest value of φ^* with respect to the position of the points S and T.

4 DYNAMIC ANALYSIS

The structure oscillates between two mechanisms with centres A’ and A’’ (Fig. 3), being φ and ψ the lagrangian parameters of the motion.

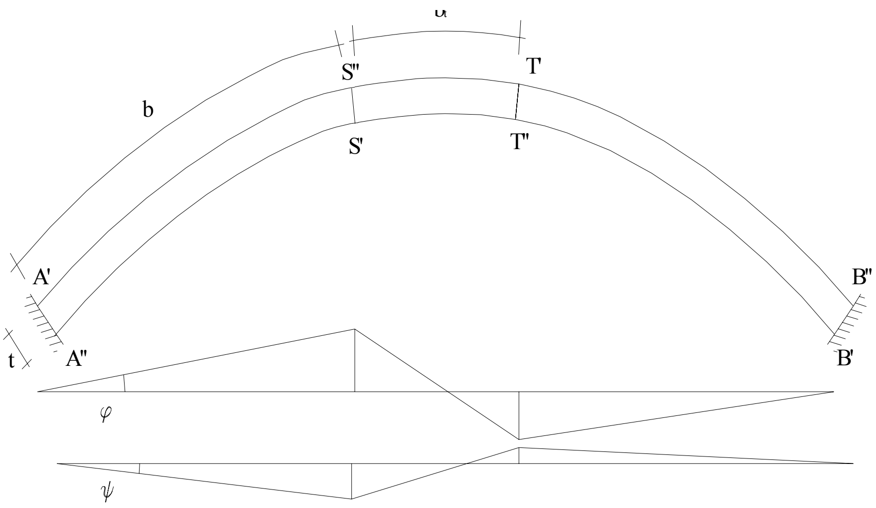

Figure 3 : The lagrangian parameters

The kinetic energy is given from

$$T_c = \frac{1}{2}\Big[\left(mr_1\dot{\varphi}\right)^2 + m_t\left(mr_2\dot{\varphi}_2\right)^2 + m\left(mr_3\dot{\varphi}_3\right)\Big] + \frac{1}{2}\left(I\dot{\varphi}^2 + I_t\dot{\varphi}_2^2 + I\dot{\varphi}_3^2\right) \tag{7}$$

where r_1, r_2, r_3 are the distances of the barycentres of the stones G1, G2 and G3 from the tree centres of the motion A, B and H, and

$$\begin{aligned} I &= \frac{m}{12}\left(b^2 + t^2\right) \\ I_t &= \frac{m_t}{12}\left(b_t^2 + t^2\right) \end{aligned} \tag{8}$$

In the approximation of first order it is possible to write:

$$\begin{aligned} \varphi_2 &= C_2\varphi \\ \varphi_3 &= C_3\varphi \end{aligned} \tag{9}$$

where

$$C_2 = \frac{z_A - z_S}{z_H - z_S} \tag{10}$$

$$C_3 = C_2\frac{z_H - z_S}{z_B - z_T} \tag{11}$$

Substituting the lasts expressions in the equation (7) we obtain

$$T_c = \frac{\dot{\varphi}^2}{2}\left(mr_1^2 + m_tC_2^2r_2^2 + mC_3^2r_3^2 + I + C_2^2I_t + C_3^2I\right) \tag{12}$$

Furthermore, with an approximation of first order, the vertical displacement of the barycentres of the three stone can be written as

$$\begin{cases} v_1 = a_1\varphi \\ v_2 = a_2\varphi \\ v_3 = a_3\varphi \end{cases} \tag{13}$$

where

$$\begin{cases} a_1 = z_A - z_{G1} \\ a_2 = C_2\left(z_H - z_{G2}\right) \\ a_3 = C_3\left(z_D - z_{G3}\right) \end{cases} \tag{14}$$

Potential energy is given by

$$P = -g\varphi\left(ma_1 + m_t a_2 + ma_3\right) \tag{15}$$

Known T_c and P the Lagrange equation

$$\frac{d}{dt}\frac{dT}{d\dot{\varphi}} + \frac{dP}{d\varphi} = 0 \tag{16}$$

became

$$I_T\ddot{\varphi} = -A \tag{17}$$

Assuming

$$C = \frac{A}{I_T} \tag{18}$$

the equation (17) can be written as

$$\ddot{\varphi} = -C \tag{19}$$

from which

$$\varphi = -C\frac{t^2}{2} + D_1 t + D_2 \tag{20}$$

Assuming the initial condition

$$\begin{cases} \varphi(0) = \varphi_0 \\ \dot{\varphi}(0) = 0 \end{cases} \tag{21}$$

it is possible to write:

$$\varphi = -C\frac{t^2}{2} + \varphi_0 \tag{22}$$

the φ function begin zero when the time t assumes the value:

$$t_1 = \left(2\frac{\varphi_0}{C} \right)^{\frac{1}{2}} \quad . \tag{23}$$

The equation (22), for $t \in [0, t_1]$, is represented in Fig. 4 (curve QH); for $t > t_1$ the centre of rotation become A'', B'', C'' and D'' so ψ become the lagrangian parameter of the motion.

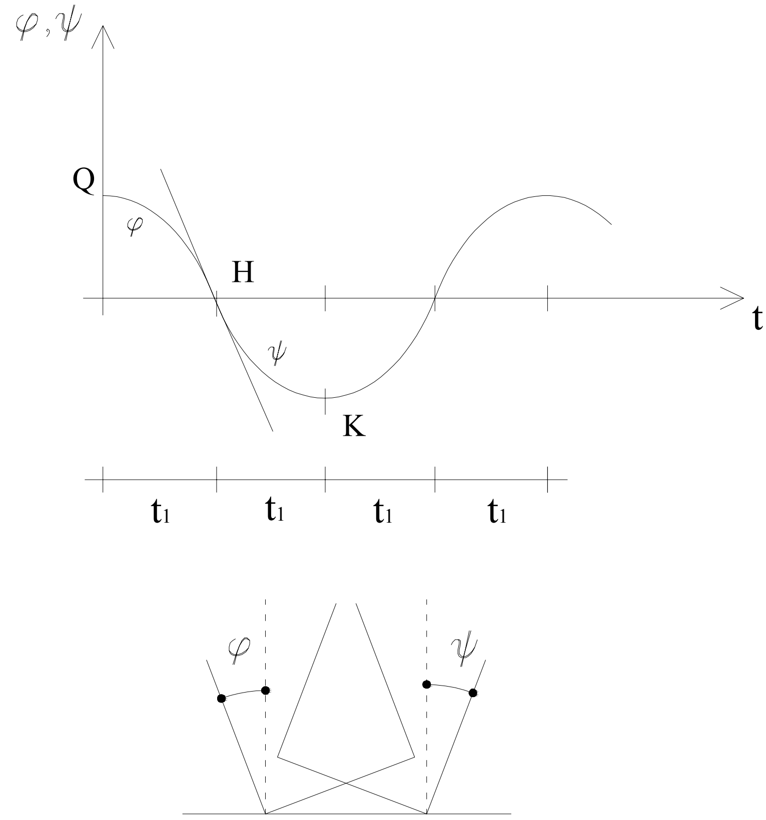

Figure 4 : Rappresentation of function φ and ψ

Assuming the conservation of energy, the equation of the second mechanism is the equation (19) but the initial conditions are:

$$\begin{cases} \psi(0) = 0 \\ \dot{\psi}(0) = V \end{cases} \tag{24}$$

and the integral of the equation (19) is

$$\psi = -C\frac{t^2}{2} + Vt \tag{25}$$

The equation (25), for $t \in [0, t_1]$, is represented in Fig. 4 (curve HI). From Fig. 4 it is possible to observe that the motion is periodic and its period is

$$T = 4t_1 = 4\left(2\frac{\varphi_0}{C} \right)^{\frac{1}{2}} \tag{26}$$

It is possible to represent the function T with respect to the ratio between the weight of the central stone and the weight of the lateral blocks m_t / m (Fig. 5)

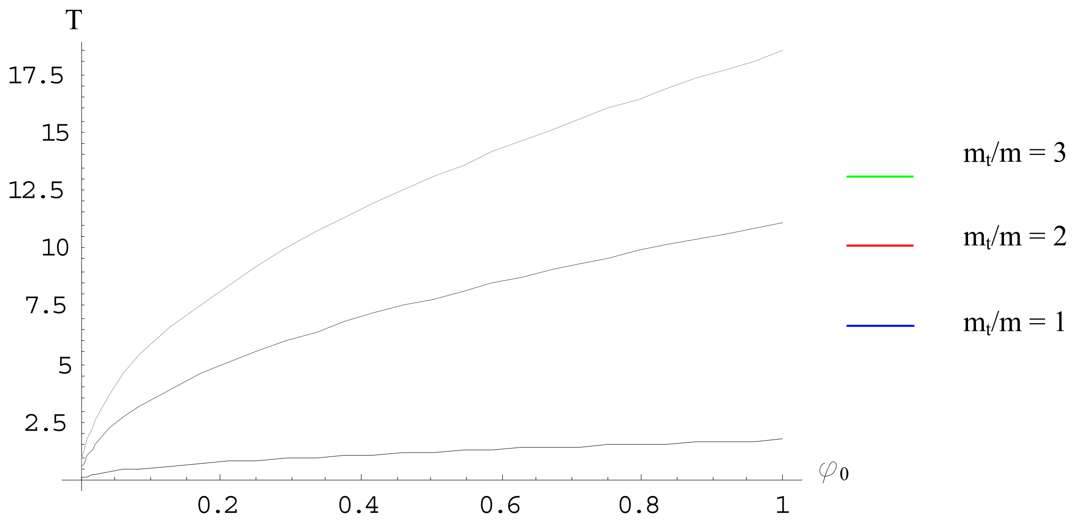

Figure 5 : Function T(φ_0)

Furthermore we have drown the T function with respect to the height of the arch h (Fig. 6).

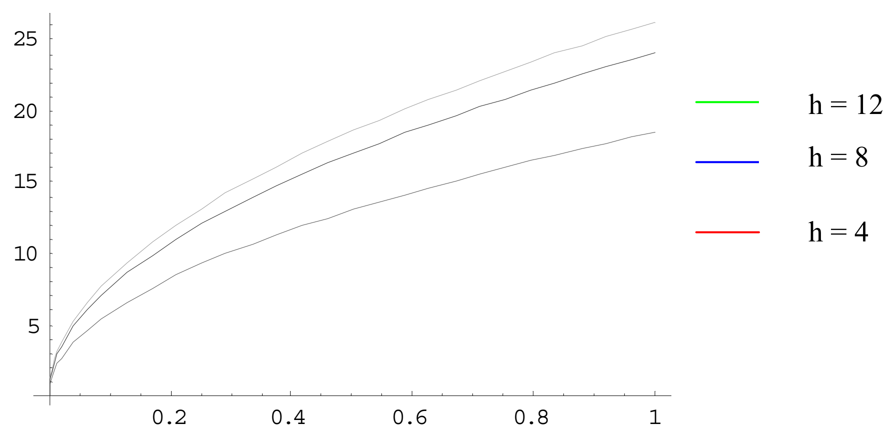

Figure 6 : Function T(h)

Of course the period T increase with the value of the ratio m_t/m and with the height h.

REFERENCES

V. Franciosi, Sul grado di stabilità degli archi monumentali - Restauro 1985
V. Franciosi, A.Sinopoli – The stability degree of masonry structures monumentali Restauro 1985
V. Franciosi, L'attrito nel calcolo a rottura delle muarture, Giornale del Geniio civile,215-234 (1980).
M.R. Migliore, F. Ramundo, E. Ruocco - Analysis of stability in monumental stone arches. STREMAH 2003. 7-9 May, Halkidiki, Greece. WIT press.

M.R. Migliore, V. Minatolo, Sulla sicurezza e stabilità delle strutture monodimensionali a blocchi, Aimeta 1995
J. Heyman, The stone skeleton, University of Cambridge UK – Int.J.Solids Structures n.2/1966
J. Heyman, The safety of masonry arches – Int. J. Mech. Sci. n.11/1969
H. Leipholz, Stability of elastic systems, Sijthoff e Nootdhoff, 1980.
S. M. Holzer, Degree of Stability of equilibrium, J. Struct. Mech.3, 61-75 (1974).