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Limit Analysis of Three-Dimensional Masonry Structures

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ABSTRACT: The paper focuses its attention on the limit analysis of masonry brick-block system. In this regard, masonry structures are modeled as a set of rigid blocks linked by interface elements having frictional (non-associative) and tensionless behavior. The formulated mathematical programming problem with equilibrium constraints is solved by a sequence of linear mathematical programming problems. The procedure is applied with reference to two and three-dimensional models already considered and studied by other researchers.

1 INTRODUCTION

The regular texture of many masonry structures has often induced researchers to the analysis of discrete models constituted by parallelepipedal blocks mutually connected by horizontal mortar joints. To this end, the uncertainty of geometric and mechanical characteristics of both blocks and mortar joints, the awareness of their difficult mechanical modeling as well as that of the computational complexity of accurate inelastic analyses have from a long time now indicated limit analysis as a valid and appropriate mathematical tool for the study of the collapse behavior of such systems.

First applications of limit analysis to masonry structures constituted by rigid blocks are reported in some papers of the half of the past century with reference to masonry arches. Nevertheless, only in the seventies the formulation of limit analysis of masonry structures is expressed in the standard form of a linear programming (LP) problem. In particular, Livesley (1978), as a principal author of such a passage, addresses the study of masonry arches under the hypothesis of contact interfaces characterized by the classical Coulomb friction law and associative flow rule. He perceives that the latter hypothesis introduces an evident inconsistency between the mathematical model and the physical problem. Nevertheless, he does not manage to modify the proposed mathematical model so as to take account of a non-associative type of contact interface law. The correct modeling of the interface behavioral law is present in a successive paper by Lo Bianco and Mazzarella (1985). These researchers properly formulate the problem of limit analysis of block masonry structures and propose a mathematical strategy aiming at the solution of the resulting non-linear programming (NLP) problem. More recently, many other researchers, e.g. Baggio and Trovalusci (1993, 1995, 1997, 1998, 2000, 2002), Fishwick (1994, 1996, 1997, 2000), Begg (1994, 1995), Ferris and Tin-Loi (2001), have focused their attention on limit analysis of block masonry. They have proposed different formulations of the non-linear programming problem and accordingly different solving methodologies of the mathematical problem. All of them have demonstrated to successfully solve plane as well as three-dimensional structures, even if under stringent behavioral hypotheses or by means of questionable solving methodologies. Owing to this, a new technique of solution of NLP problems is presented here which (1) requires the solution of only LP problems, (2) allows the consideration of yielding laws different from the classical Coulomb one as well as (3) the simple and accurate application to three-dimensional block structures.

2 FORMULATION OF LIMIT ANALYSIS OF RIGID BLOCK MASONRY

For the sake of simplicity, in this present paper the mathematical problem is formulated in detail with reference to two-dimensional schemes. Some notes are, anyway, reported at the end aiming at explaining how easily and correctly the proposed methodology of solution of NLP problems is extended to three-dimensional structures.

Masonry is modeled by means of rigid blocks interacting by means of interface elements able to transfer unlimited compressive stresses but prevented from resisting tension. Interfaces are, furthermore, assumed to be able to transfer shear stresses in virtue of the presence of friction as well as of the interlocking phenomenon between opposite faces of adjacent blocks. As shown in Fig. 1 the possibility of development of non-uniform distributions of normal stresses justifies the capacity of elements to transmit axial forces as well as bending moments.

Kinematics of single blocks is characterized by rigid translations along two orthogonal axes and by rotation around the centroid of mass of the block. Mutual internal actions among blocks are evaluated in terms of stress resultants. Under such hypotheses, the stress resultants give rise to axial forces *n*, shear forces *t* and bending moments *m*. Their positive sign is shown in Fig. 1. Loads are distinguished between fixed and live loads and applied at the centroid of mass of the single block.

With reference to such a model equilibrium equations are expressed as (Ferris and Tin-Loi, 2001)

$$\mathbf{Ax} = \mathbf{f}_D + \alpha \mathbf{f}_L \tag{1}$$

where A represents the equilibrium matrix, x the vector of the stress resultants, f_D the vector of known dead loads, αf_L the vector of unknown live loads and α an unknown proportional load factor that amplifies the known vector of basic live loads. The equilibrium matrix is of order 3b x 3c being b the number of blocks and c that of interfaces.

In the following kinematic quantities (displacements and strains) are assumed to be in rate form and are denoted, for clarity, without the normal superscript. Compatibility equations establish a mathematical relationship between the vector u of displacement rates of blocks and the vector q of interface strain rates as follows:

$$\mathbf{q} = \mathbf{A}^{\mathbf{T}}\mathbf{u} \tag{2}$$

Interface resultants are possible if stress points are inside limit yield surfaces which take into account of both sliding and rocking. In particular, stress resultants identified by points of the limit yield surface produce interface plastic deformations due to either sliding or rocking. In the hypothesis of a classical Coulomb friction law the limit yield conditions are described by the following relationships:

$$\begin{bmatrix} y_{iS+} \\ y_{iS-} \\ y_{iR+} \\ y_{iR-} \end{bmatrix} = -\begin{bmatrix} -\sin\phi_i & \cos\phi_i & 0 \\ -\sin\phi_i & -\cos\phi_i & 0 \\ -\sin\psi_i & 0 & \cos\psi_i \\ -\sin\psi_i & 0 & -\cos\psi_i \end{bmatrix} \begin{bmatrix} n_i \\ t_i \\ m_i \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

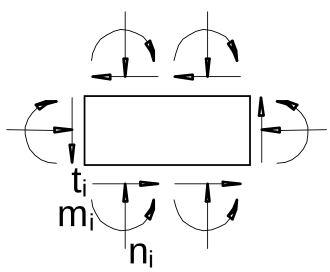


Figure 1 : Stress resultants applied to the single block

where y is a vector of orthogonal distances from the stress point at the generic interface to the limit yield surface, ϕ_I the angle the tangent of which coincides with the Coulomb coefficient and ψ_I the angle the tangent of which coincides with half of the block length. Finally, in such an equation sub-indices S and R refer to yield surfaces due to sliding and rocking, respectively.

The same relationships may be presented more compactly as:

$$\mathbf{y}^i = -\mathbf{N}^{i\mathrm{T}}\mathbf{x}^i \geq \mathbf{0} \tag{4}$$

Accordingly, the limit yield conditions referred to the whole structure may be formulated as:

$$\mathbf{y} = -\begin{bmatrix} \mathbf{N}^{1\mathrm{T}} & 0 & \dots & 0 \\ 0 & \mathbf{N}^{2\mathrm{T}} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathbf{N}^{c\mathrm{T}} \end{bmatrix} \mathbf{x} = -\mathbf{N}^{\mathrm{T}}\mathbf{x} \geq \mathbf{0} \tag{5}$$

The vector of strain rates of the single interface is linked to the nonnegative resultant strain rates of the same interface by means of the following relationships:

$$\begin{bmatrix} \gamma \\ \varepsilon \\ \theta \end{bmatrix} = \begin{bmatrix} \cos\phi_o & -\cos\phi_o & 0 & 0 \\ -\sin\phi_o & -\sin\phi_o & -\sin\psi & -\sin\psi \\ 0 & 0 & \cos\psi & -\cos\psi \end{bmatrix} \begin{bmatrix} z_{S+} \\ z_{S-} \\ z_{R+} \\ z_{R-} \end{bmatrix}, \quad \begin{bmatrix} z_{S+} \\ z_{S-} \\ z_{R+} \\ z_{R-} \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

or in a more compact form:

$$\mathbf{q}^{\mathbf{i}} = \mathbf{V}^{\mathbf{i}}\mathbf{z}^{\mathbf{i}}, \ \mathbf{z}^{\mathbf{i}} \geq \mathbf{0} \tag{7}$$

Complementarity equations, are instead expressed by means of the relationship:

$$\mathbf{y}^T\mathbf{z} = 0 \tag{8}$$

They state the consistency between static and kinematic variables and indicate that plastic strain rates are possible only if interfaces are in the incipient collapse configuration.

Finally, live loads must cause positive work to be produced by the displacement rates u. Such a condition may be written in the following form:

$$\mathbf{f}_{\mathbf{L}}^{\mathbf{T}}\mathbf{u} = \mathbf{1} \tag{9}$$

3 CLASSICAL FORMULATION OF THE MATHEMATICAL PROBLEM UNDER THE HYPOTHESIS OF COULOMB FRICTION LAW

If interface elements are supposed to be characterized by associative laws either static or kinematic theorems may be applied. If static approach is used the problem may be written as

$$\begin{aligned} &\text{maximize} && \alpha \\ &\text{subject to} && -\alpha\mathbf{f}_{\mathrm{L}} + \mathbf{A}\mathbf{x} = \mathbf{f}_{\mathbf{D}} \\ & && -\mathbf{N}^{\mathrm{T}}\mathbf{x} \geq \mathbf{0} \end{aligned} \tag{10}$$

In this case, the solution of the mathematical problem is constituted by the maximum value of the live load multiplier which satisfies equilibrium equations and limit yield conditions.

If kinematic approach is followed, instead, the problem may written as

$$\begin{aligned} &\text{minimize} && -\mathbf{f}_\mathbf{D}^\mathbf{T}\mathbf{u} \\ &\text{subject to} && \mathbf{f}_\mathbf{L}^\mathbf{T}\mathbf{u} = \mathbf{1} \\ &&& -\mathbf{A}^\mathbf{T}\mathbf{u} + \mathbf{N}\mathbf{z} = \mathbf{0} \\ &&& \mathbf{z} \geq \mathbf{0} \end{aligned} \tag{11}$$

Problems stated by Eqs. (10-11) are dual and presume an associative type of contact interface law. Unfortunately, reality shows that such a hypothesis is not true. Indeed, an associative law implies the phenomenon of dilatancy, i.e. the development of translation rates orthogonal to rigid block faces.

In the absence of dilatancy the division of the problem of limit analysis into two LP problems founded on either static or kinematic theorems is not possible. The mathematical problem may then be stated in the following form:

$$\begin{aligned} &\text{minimize} && \alpha \\ &\text{subject to} && \mathbf{f}_\mathbf{L}^\mathbf{T}\mathbf{u} = \mathbf{1} \\ &&& -\mathbf{A}^\mathbf{T}\mathbf{u} + \mathbf{V}\mathbf{z} = \mathbf{0} \\ &&& -\alpha\mathbf{f}_\mathbf{L} + \mathbf{A}\mathbf{x} = \mathbf{f}_\mathbf{D} \\ &&& \mathbf{y} = -\mathbf{N}^\mathrm{T}\mathbf{x} \\ &&& \mathbf{y} \geq \mathbf{0} \quad \mathbf{z} \geq \mathbf{0} \quad \mathbf{y}^\mathbf{T}\mathbf{z} = \mathbf{0} \end{aligned} \tag{12}$$

This problem returns the collapse mechanism characterized by no dilatancy at the cost of high mathematical difficulties owing to the non-linearity of the problem. Still with reference to the type of contact interface law it should be added that the hypothesis of associative law imposes N=V, while that of non-associative law implies N≠V. Finally, an associative law is always adopted with reference to the flow law due to rocking.

4 ITERATIVE METHOD

The proposed method for the solution of NLP problems applied to block masonry involves the iterative solution of LP problems. Non-associative flow rules are considered with reference to limit yield surfaces for either rocking or sliding. But, while proper yield surfaces are adopted for rocking, fictitious yield conditions are assumed with reference to sliding. Indeed, at each step of the iterative procedure a limit yield surfaces for sliding is adopted which states the static admissibility of shear forces limited by prefixed minimum and maximum values:

$$t_i^l \leq t_i \leq t_i^u \tag{13}$$

Introduction of the new limit yield surface for sliding determines modifications of some relationships previously reported. Indeed, limit yield conditions referred to the single interface are now formulated as:

$$\begin{bmatrix} y_{iS+} \\ y_{iS-} \\ y_{iR+} \\ y_{iR-} \end{bmatrix} = -\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ -\sin\psi_i & 0 & \cos\psi_i \\ -\sin\psi_i & 0 & -\cos\psi_i \end{bmatrix} \begin{bmatrix} n_i \\ t_i \\ m_i \end{bmatrix} + \begin{bmatrix} -t^u \\ t^l \\ 0 \\ 0 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

or, more compactly as:

$$\mathbf{y}^i = -\mathbf{V}^{i\mathrm{T}}\mathbf{x}^i + \mathbf{P} \geq \mathbf{0} \tag{15}$$

being P the vector of limit shear stress resultants. With reference to the whole structure the new limit yield conditions are, therefore:

$$\mathbf{y} = -\mathbf{V}^{\mathrm{T}}\mathbf{x} + \mathbf{P} \geq \mathbf{0}$$

$$\mathbf{y} = -\begin{bmatrix} \mathbf{V}^{1\mathrm{T}} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{V}^{2\mathrm{T}} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{V}^{m\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \vdots \\ \mathbf{x}^m \end{bmatrix} + \begin{bmatrix} \mathbf{P}^1 \\ \mathbf{P}^2 \\ \vdots \\ \mathbf{P}^m \end{bmatrix} \geq \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \tag{16}$$

Obviously, in real applications limit yield conditions for sliding are different from those referred in Eq.(13) because yield shear resultants generally depend on the normal stress resultants. Nevertheless, the adopted limit yield surface for sliding would lead to the exact load multiplier if normal stress resultants corresponding to the true collapse configuration were known. Unfortunately, the normal stress state of interface elements is not known a priori and, consequently, values t^l e t^u are unknown at the beginning of the process. In the classical case of a limit yield surface complying with a Coulomb friction law such values are function of the normal stress resultants as reported by the following relationships:

$$t_i^u = \tan\phi_i \cdot n_i \tag{17}$$

$$t_i^l = -\tan\phi_i \cdot n_i \tag{18}$$

Limits t^l e t^u being fixed at the generic step of the iterative procedure, the collapse load multiplier α is determined as solution of the LP problem:

$$\begin{aligned} &\text{max imize} && \alpha \\ &\text{subject to} && -\alpha\mathbf{f}_{\mathrm{L}} + \mathbf{A}\mathbf{x} = \mathbf{f}_{\mathrm{D}} \\ & && -\mathbf{V}^{\mathrm{T}}\mathbf{x} + \mathbf{P} \geq 0 \end{aligned} \tag{19}$$

In order to find the solution of the collapse load multiplier, an iterative method is suggested in which the n_i variables are assumed equal to those of the previous iteration. Once the collapse load multiplier has been found, the vector y is determined as:

$$\mathbf{y} = -\mathbf{V}^{\mathrm{T}}\mathbf{x} + \mathbf{P} \tag{20}$$

and the displacement rates corresponding to the collapse mechanism are attained as solution of the following mathematical LP problem:

$$\begin{aligned} &\text{min imize} && -\mathbf{f}_{\mathrm{D}}^{\mathrm{T}}\mathbf{u} + \mathbf{P}\ \mathbf{z} \\ &\text{subject to} && \mathbf{f}_{\mathrm{L}}^{\mathrm{T}}\mathbf{u} = 1 \\ & && -\mathbf{A}^{\mathrm{T}}\mathbf{u} + \mathbf{V}\mathbf{z} = \mathbf{0} \\ & && \mathbf{z} \geq \mathbf{0} \quad \mathbf{y}^{\mathrm{T}}\mathbf{z} = \mathbf{0} \end{aligned} \tag{21}$$

In this regard, we must recognize that infinite combinations of stress resultants may exist which are in equilibrium with the collapse loads. Accordingly, the stress vector x resulting from the previous steps is to be considered only as one of the possible solutions, while the solving collapse load multiplier as the unique solution of the mathematical LP problem in which limit shear values have been fixed according to Eq. (20-21) on the basis of the assumed vector of normal stress resultants.

To identify the minimum value of the collapse load multiplier, many vectors of stress resultants statically admissible and in equilibrium with the collapse loads are calculated by the following mathematical problem:

$$\begin{aligned} &\max \text{imize} && \mathbf{c} \cdot \mathbf{x} \\ &\text{subject to} && \mathbf{A}\mathbf{x} = \mathbf{f}_{\mathrm{D}} + \alpha \mathbf{f}_{\mathrm{L}} \\ & && -\mathbf{V}^{\mathrm{T}}\mathbf{x} + \mathbf{P} \geq 0 \end{aligned} \tag{22}$$

where c is a vector of known coefficients. In order to reach a reliable estimate of the exact collapse load multiplier, many vectors c are randomly defined prior to the beginning of the iterative procedure. Obviously, the exact solution is that one characterized by the lowest value of the load multiplier.

5 LIMIT ANALYSIS OF THREE-DIMENSIONAL BLOCK STRUCTURES

When dealing with three-dimensional structures an orthonormal local basis has to be defined with reference to the single interface element (Fig. 2). The local coordinate system is right-hand and origins at the centroid of interfaces. For the sake of simplicity, 2 and 3-axes are supposed as parallel to the sides of interfaces. Positive sign of both interface stress resultants and external forces of the single blocks are reported in Fig. 3.

Differently from what has been stated with reference to two-dimensional structures the interface normal stress resultants of three-dimensional systems must satisfy the conditions:

$$N_i \geq 0 \tag{23}$$

$$\sqrt{V_{2i}^2 + V_{3i}^2} \leq N_i \tan \phi_i \tag{24}$$

$$|M_{2i}| \leq N_i \frac{l_{3i}}{2} \tag{25}$$

$$|M_{3i}| \leq N_i \frac{l_{2i}}{2} \tag{26}$$

where, as shown in Fig. 2, l_2 is the length of segment 2-3 of interfaces and l_3 is the length of segment 1-2. All the relationships are linear apart from Eq.(24). Nevertheless, such an equation may be substituted by other two linear relationships:

$$V_2 \leq \mu N \sin \varphi \tag{27}$$

$$V_3 \leq \mu N \cos \varphi \tag{28}$$

in which $\varphi = \arctan V_2 / V_3$ represents the slope of the shear force with respect to the 3-axis.

The limit values of the torque have been simply identified as a proportion of the axial force by means of a fictitious coefficient of friction, as:

$$M_{1i}^l \leq M_{1i} \leq M_{1i}^u \tag{29}$$

where :

$$M_{1i}^l = -\mu_t \cdot N_i \quad \text{and} \quad M_{1i}^u = \mu_t \cdot N_i \tag{30}$$

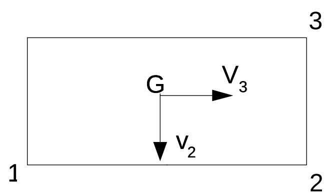


Figure 2 : Orthonormal local basis

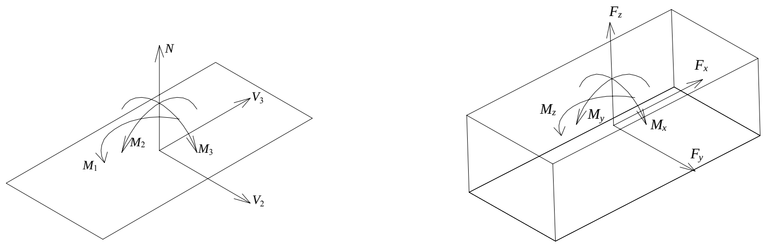


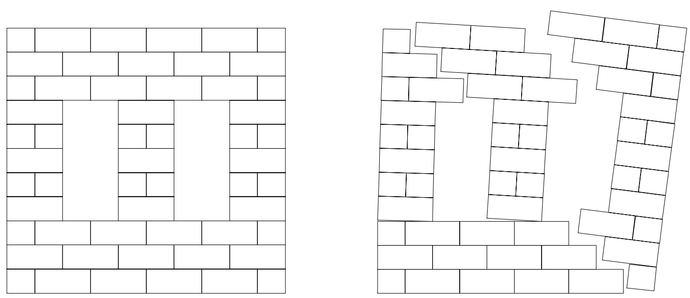
Figure 3 : Positive sign of interface stress resultants and external forces of single blocks

6 NUMERICAL APPLICATIONS

The proposed method has been applied to two-dimensional as well as to three-dimensional block structures (Pantano, 2003). In particular, several two-dimensional models have been tested by mean of the proposed mathematical formulation and solving methodology of the corresponding NLP problem. In all cases the results are in good agreement with those of other authors. Nevertheless, for lack of space only two experiments are shown here. The models (Figs. 4-5) are constituted by two dimensional structures analyzed in the past by Baggio and Trovalusci and by Ferris and Tin-Loi (2001). Their collapse configuration is confirmed by the laboratory experiment and, as reported at the bottom of the Figure, characterized by a collapse load multiplier equal or just a little higher than that predicted by Ferris and Tin-Loi (2001). The absence of dilatancy slightly decreases the value of the collapse load multiplier. Obviously, this depends on the modest presence of sliding of blocks in the collapse configuration. The present results are obtained after fifteen iterations by means of ten different vectors **c**. Finally, even not shown in any figure the proposed model has been applied to a simple three-dimensional model constituted by two orthogonal small masonry panels. The analysis has highlighted the reliability of the method and the importance of even small values of the friction coefficient referred to torque for the convergence of analysis. Further details may be found in Pantano (2003).

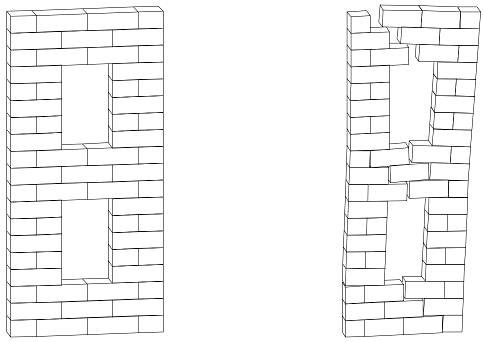
7 CONCLUSIONS

The paper describes an iterative solving methodology for non-linear programming problems deriving from the limit analysis of block masonry structures. Differently from other strategies the present procedure allows the solution of non-linear programming problem by means of the solution of linear programming problems.



Block number	Interface number	With dilatancy	Without dilatancy	
			Proposed method	Tin Loi
55	116	0.33195	0.26374	0.26374

Figura 4 : Numerical application to a two-dimensional model (model 1)



Block number	Interface number	With dilatancy	Without dilatancy	
			Proposed method	Tin Loi
61	120	0.23964	0.21964	0.20863

Figure 5 : Numerical application to a two-dimensional model (model 2)

The solving methodology easily allows the consideration of yield surfaces different from that complying with classical Coulomb friction law. Furthermore, it allows a proper consideration of the limit yield condition imposed to the interface shear stress resultants. Nevertheless, some further attention has to be focused on the limit yield condition referred to torque. Numerical applications referred to both two and three-dimensional models highlight the appropriateness of the mathematical formulation and the reliability of the methodology of solution of non-linear programming problems.

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