Structural Analysis of Historical Constructions, New Delhi 2006
P.B. Lourenço, P. Roca, C. Modena, S. Agrawal (Eds.)

On Drilling DOF's of Membrane Elements and Application to Historical Structures

Andres Mena
Laboratory for Structural Engineering, Swiss Federal Laboratories for Materials Testing and Research (EMPA), Dübendorf, Switzerland

Yasin M. Fahjan
Department of Earthquake and Structural Science, Gebze Institute of Technology, Kocaeli, Turkey

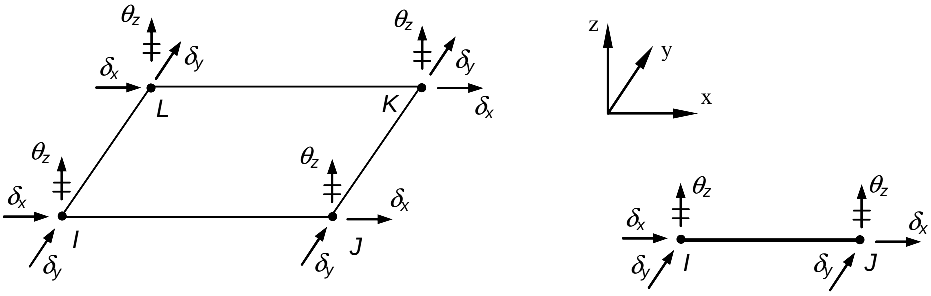
ABSTRACT: This study investigates the performance of membrane elements, particularly in the 2D analysis of historical structures. Two formulations of the membrane element in the linear elastic framework and one formulation in the geometrically nonlinear framework are presented. A plasticity model is considered in the analysis as well. The three membrane formulations are employed to analyze a cross section of a historical structure. The effect of drilling DOF's and incompatible displacement modes in the analytical results is discussed.

1 INTRODUCTION

The analysis of historical structures is particularly complex in several aspects. For example, difficulties exist in available geometric data, material characterization, reliable numerical tools, etc. Concerning the finite element modeling for historical structures, different strategies for the analysis are applicable, depending on resources, detail requirements and ability of the practitioners. The massive structure of historical structures is generally unreinforced masonry. When unreinforced masonry is treated as a composite in numerical models the analytical strategy employed is known as macro-modeling, Lourenço (2002). In macro-modeling, units, mortar and the interface between mortar and units are smeared out in a homogenous continuum. No independent properties of the individual components of unreinforced masonry are necessary. Instead, a relation between average stresses and strains of the composite is established. Consequently, macro-modeling is more practice oriented due to reduced time and memory requirements as well as a user-friendly mesh generation. This type of modeling is most valuable when a compromise between accuracy and efficiency is needed.

Although effort has been spent in proposing material models for unreinforced masonry in the macro-modeling strategy, less attention has been given to the element formulation itself. In the 2D in-plane finite element formulation, standard membrane elements contain two translational degrees of freedom per node (plane stress element). The addition of rotational degrees perpendicular to the element surface has been a challenge for researchers. Membrane elements including drilling DOF's are of great practical interest. When connected to frame elements, membrane elements of this type fully transfer the rotational stiffness providing consistency and versatility to the finite element modeling (see Fig. 1). The search for a 4-node membrane element with normal rotations (drilling rotations) was a fruitless endeavor for the first thirsty years of the development of finite element technology. Nevertheless, theoretical approaches have evolved in the direction of a reliable membrane element of this type in the frameworks of geometrically linearity and nonlinearity.

In linear elasticity, numerous works have appeared in the engineering research literature in which successful approaches that includes drilling degrees of freedom in membrane elements have been described. Examples are the works of Ibrahimbegović *et al* (1990), Ibrahimbegović and Wilson (1991), Ibrahimbegović and Frey (1992) and Geyer and Groenwold (2002). In geometrically nonlinearity, the complexity is remarkable respect to the linear framework. The work of Ibrahimbegović and Frey (1995) presented very robust formulation by providing a sound variational formulation valid directly for geometrically nonlinear theory and applicable to 4-node membrane with independent rotations. Rotations and displacements are interpolated with standard shape functions. Other successful works in this topic are presented by Ibrahimbegović (1993), Ibrahimbegović and Frey (1993b) and Yongi and Zacharia (1996). Extensively, the formulations of the membrane element can be employed to accommodate a suitable plasticity model. Lourenço *et al*. (1997) presented a plane stress softening plasticity model for quasi-brittle orthotropic materials, which has shown successfulness in representing the orthotropic material behavior of unreinforced masonry.



Membrane element with corner drilling rotations

Planar frame element

Figure 1 : Degrees of freedom in membrane and planar frame elements.

In this study, we present and validate three formulations of the membrane element; we investigate their performance in the linear elastic and material nonlinear analyses of a historical structure, emphasizing the effect of drilling DOF's and incompatible displacement modes. For the computational implementation, homemade MatLab codes have been developed. In Mena (2006), more details about the membrane formulations are given.

2 THEORY

2.1 Plane stress element including incompatible modes

The simple 4-node plane stress element with translational corner displacements does not produce accurate results for many applications, Wilson and Ibrahimbegović (1990). It is enough to apply a pure bending load to a single plane stress element with a minimum number of constraints, to show that the resultant stresses are not correct. This problem is known as shear locking. To overcome the shear locking, incompatible displacement modes are added to the plane stress formulation,

$$N_5 = 1 - r^2 ,\ N_6 = 1 - s^2 \tag{1}$$

The incompatible modes are directly associated with four artificial displacements α that enlarge the strain-displacement matrix $\boldsymbol{B}$.

$$[\varepsilon] = \left[B_c B_i\right] \begin{bmatrix} u \\ \alpha \end{bmatrix} \tag{2}$$

In order to eliminate the strain energy associated with the incompatible modes, a correction is added to the $\boldsymbol{B_i}$ matrix. Afterwards, in the element stiffness matrix, the terms arising from the incompatible displacements are eliminated by static condensation. This methodology has demonstrated effectiveness for plane rectangular elements even with distorted mesh. Our implementation of the linear plane stress element with incompatible modes has been denoted LPSIM.

2.2 A geometrically linear framework of the membrane element

A robust quadrilateral 4-node membrane element containing drilling DOF's was developed by Ibrahimbegović *et al* (1990). In that study, the variational formulation is given in two different forms: mixed-type formulation and displacement-type formulation. Both of them give equivalent results. In our study, we follow the displacement-type. The strain-displacement equations of the membrane element can be written in sub matrix form,

$$[\varepsilon] = [B \ G] \begin{bmatrix} u \\ \theta \end{bmatrix} \tag{3}$$

The ***B*** matrix has the standard form, that is, the strain-translational displacement relationship. ***G*** is the strain-displacement field corresponding to drilling rotations. Displacements and rotations are interpolated by using the standard and serendipity shape functions. The correction using incompatible modes assures that the element satisfies the constant stress patch test. A hierarchical bubble function interpolation is also added to the displacements, and its contribution to the element stiffness matrix is eliminated by static condensation. The element stiffness matrix is computed using 3x3 Gaussian quadrature. To remove the spurious zero-energy mode due to non-conventional interpolation of rotations, a penalty term is added to the element stiffness matrix. The penalty function is evaluated by a single point Gaussian quadrature. We have named this element LMR4.

2.3 A geometrically nonlinear framework of the membrane element

The formulation of Ibrahimbegović and Frey (1995) in the nonlinear framework of a 4-node membrane element with drilling DOF's is utilized. As in the linear theory, mixed-type and displacement-type formulations are presented. Both formulations are valid. In matrix notation, the displacement-type variational formulation is defined as,

$$\Pi_{\gamma}(u,\theta) = \int_V \left\{ 0.5 e^T(u,\theta) D e(u,\theta) + 0.5\gamma(\omega(u,\theta))^2 \right\} dV - \int_V u^T f \, dV \tag{4}$$

which is a simple expression in terms of computational effort. Displacements and rotations are interpolated with the standard isoparametric shape functions. The directional derivative of the variational expression in Eq. 4 gives the virtual work expression. Using the Newton's method, the residual is retained and a new directional derivative of the virtual work expression gives rise to the tangent stiffness matrix. In the solution of the nonlinear equations, we have implemented the modified Newton-Raphson method. The tangent stiffness matrix is evaluated using 2x2 Gaussian quadrature. We have named this element NLMR4. In this element, no incompatible displacement modes have been added.

2.4 Material nonlinearity: plane stress plasticity model for orthotropic unreinforced masonry

The orthotropy of masonry is well represented by the plasticity model introduced by Lourenço et al. (1997). A composite yield criterion is proposed; combining concepts of modern plasticity with anisotropic material behavior. The material model features a Hill-type criterion for compression and a Rankine-type criterion for tension. In both directions, different hardening/softening behaviors along each material axis are included. The elastoplastic equations for single and multisurface plasticity are formulated in the frame of an implicit Euler backward algorithm. They are solved using a regular Newton-Raphson method. The updated tangent stiffness matrix of the material is retained at the end of the return mapping solution to compute the tangent stiffness matrix at the element level (integration points).

3 NUMERICAL EXAMPLES

In order to validate the three membrane formulations, basic examples are numerically evaluated. Afterwards, one cross section of a historical structure is studied.

3.1 Basic examples

We have checked that the formulations pass a patch test performed on a one-element test. A skewed element is fixed with a minimum number of constraints and exposed to uniform tension (see Fig. 2). All drilling degrees of freedom are left unconstrained. The single element test gives the correct results in the three formulations.

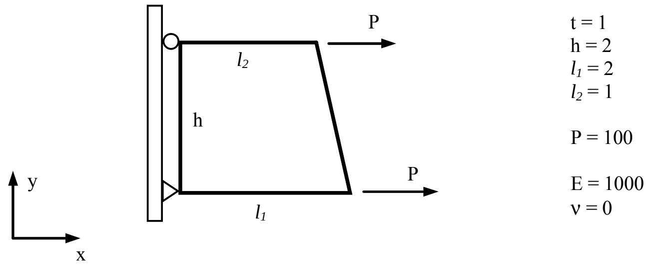


Figure 2 : One element test

Two further tests of the one-element are in the framework of the plasticity model. The ability of the model to represent tensile and compressive orthotropic behaviors is separately studied. To represent tensile behavior, the material properties assigned to the one element are: $E_x = 10000$, $\nu = 0.2$, tensile strength $ft_x = 1$ and fracture energy $G_{ftx} = 0.02$. In the case of uniform compression, the material properties are: compressive strength $fc_x = -10$ and fracture energy $G_{fcx} = 5$. The material properties in y-direction are half of those in x-direction. The single element is squared of side 100. Analytical results are shown in Fig. 3 in terms of horizontal and vertical Stress-Strain relationships. The implemented plasticity model correctly describes the material behavior. In the two examples, the three formulations predict the behavior correctly.

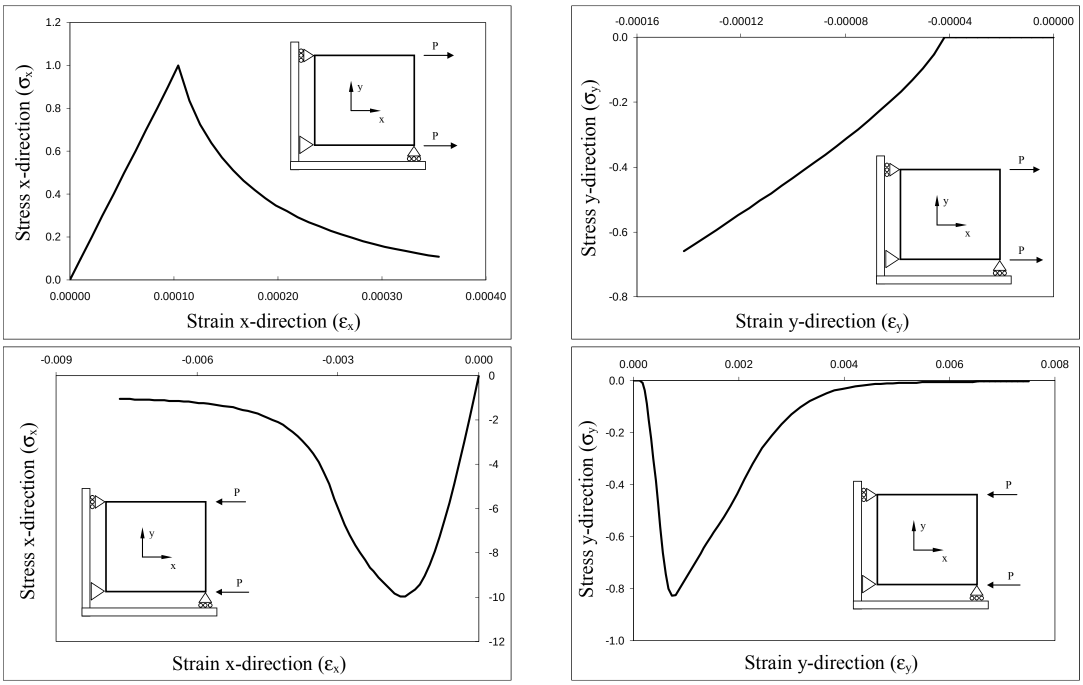


Figure 3 : One element test under uniform tension and compression

The performance of NLMR4 in the geometrically nonlinear framework is next validated. The numerical example is a straight cantilever beam of a unit length, and a height h = 0.1 under a vertical force P = 100 applied at the free end. The cantilever is of unit thickness and elasticity modulus E = 120000. The solution is obtained with a mesh of ten NLMR4 and is expressed in terms of the free-end displacement components. It is compared with the corresponding results of the membrane element QR4 of Ibrahimbegović and Frey (1995). Results are shown in Table 1.

NLMR4 has slight difference respect to QR4. In addition, the exact solution obtained with ten beam elements given by Ibrahimbegović and Frey (1993a) is shown.

Table 1 : Free-end displacement components.

Model	Horizontal	Vertical	Rotation
NLMR4	-0.415	0.748	1.245
QR4	-0.388	0.722	1.271
Beam (exact result)	-0.555	0.811	1.430

3.2 A cross section of The Fatih Mosque

The following analysis is performed on a cross section of The Fatih Mosque of Istanbul. The Fatih Mosque was built between 1463 and 1470. It was restored following an earthquake in 1509. The dome of the mosque collapsed in the 1766 earthquake and was reconstructed entirely in 1771. The Fatih Mosque has been affected by several strong earthquakes. The mosque is located on soft soil, which provokes significant amplification of the earthquake response on the structure. This cross section is one of the most important carrying load sections of the structure. The main dome and other smaller domes lay on this cross section. The materials existing in the cross section are different type of stones and solid bricks. The thickness of the wall is non-uniform as well. Due to these uncertainties, we selected constant average thickness and material that can appropriately represent the wall properties. In the finite element mesh, all degrees of freedom along the bottom are restrained. For the entire mesh, the thickness is 2.6 m. Fig. 4 shows the geometrical properties of the cross section.

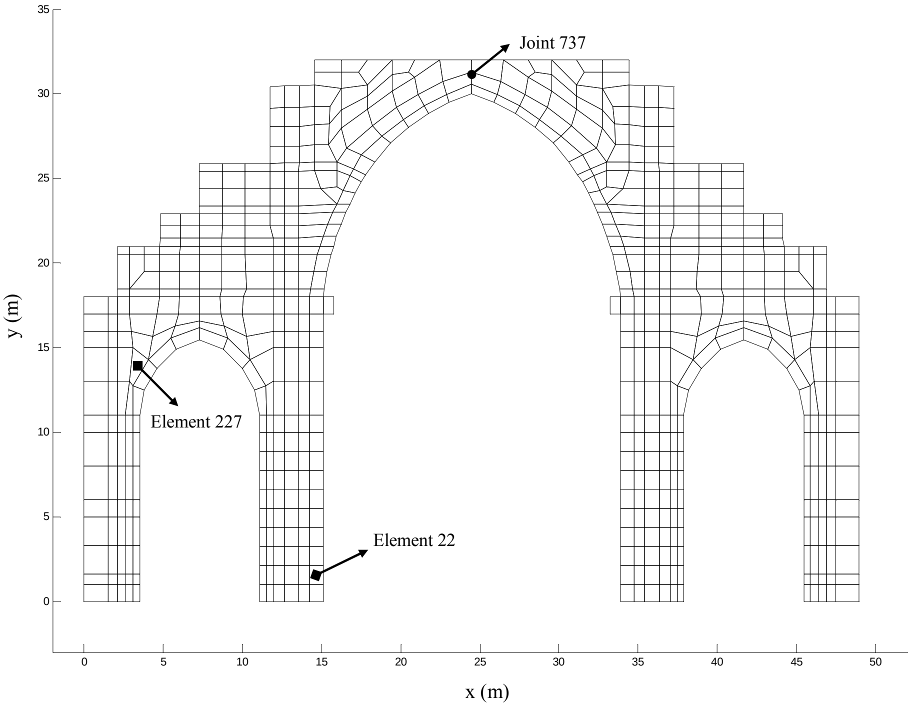


Figure 4 : Geometrical properties of the cross section of The Fatih Mosque

Material properties of the cross section in the elastic regime are: E_x=1.5e6 kN/m^2, E_y=3e6 kN/m^2, ν_{xy}=0.25 and G_{xy}=8e4 kN/m^2; in the tension regime are: f_{tx}=2e3 kN/m^2, f_{ty}=1e3 kN/m^2, α=1.5, G_{fx}=2e3 kN-m/m^2 and G_{fy}=1e3 kN-m/m^2 and in the compression regime are: f_{cx}=1e4 kN/m^2, f_{cy}=2e4 kN/m^2, β=-1.05, γ=1.2, G_{fcx}=1e4 kN-m/m^2, G_{fcy}=2e4 kN-m/m^2 and κ_p=1e-4.

A combination of the structural self-weight and earthquake forces in the form of static equivalent horizontal load was applied at every structural joint. The self-weight of the upper structure, domes, etc., is applied as distributed load along top of the model. The self-weight of the structural model itself is distributed at every joint. In the calculation of the earthquake load, a response spectrum for the characteristics of the area was built. The horizontal loads per joint are distributed following the equation,

$$F_i = \frac{m_i h_i}{\sum m_i h_i} V_b \tag{5}$$

where F_i, m_i and h_i are the horizontal force, mass and height of the joint "i" and V_b is the total base shear.

A first investigation in the linear elastic range is made. LPSIM and LMR4 formulations are utilized. Table 2 shows the displacement components at joint 737 (see Fig. 4 for joint location) corresponding to the full vertical and horizontal loads applied in one step. Both formulations agree in horizontal displacements, while LMR4 gives almost double vertical component.

Table 2 : Displacements at joint 737 (Linear elastic analysis).

Model	Horizontal (m)	Vertical (m)	Rotation
LPSIM	0.246	-0.0129	-
LMR4	0.245	-0.0247	0.0086

The material nonlinear analysis described in subchapter 2.4 is carried out with NLMR4 and LPSIM. First the total vertical load is applied in one load step and later the horizontal load in increments. Fig. 5 shows the Base shear vs. Horizontal displacements curves at joint 737. The base shear in the linear elastic solution differs of nonlinear solutions about 25%. The solution with NLMR4 is somewhat stiffer than with LPSIM. No geometrically nonlinear analysis is performed since it gives similar results when compared to the linear elastic analysis and can be neglected in the analysis of large unreinforced masonry walls, Mena and Fahjan (2005).

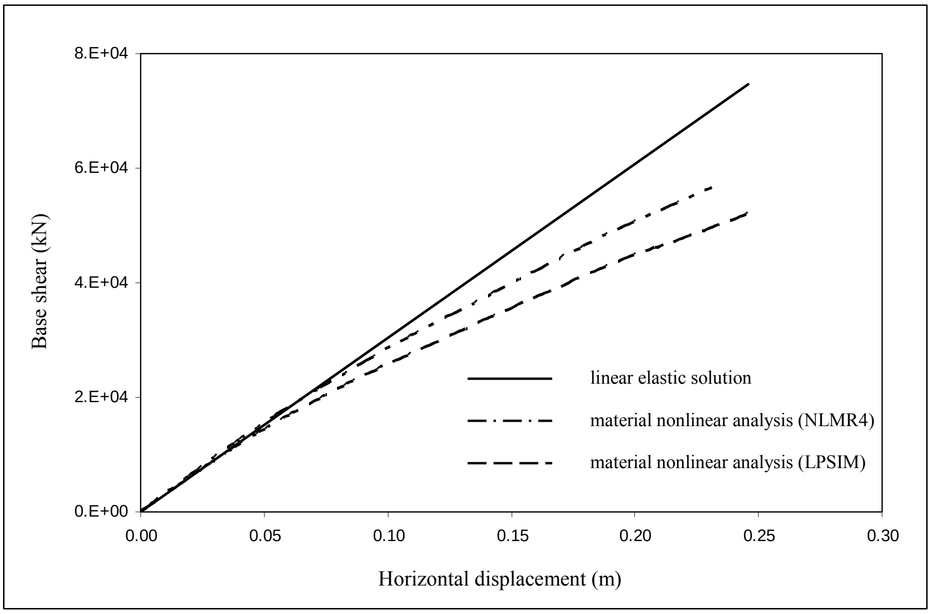


Figure 5 : Base shear vs. Horizontal displacements at joint 737

On the other hand, the plasticity model describes the softening behavior after cracking and crushing. Particularly in this cross section, degradation of the structural stiffness takes place when the tensile stress in the vertical direction reaches the maximum tensile strength. Fig. 6 represents the stress distribution in y-direction for the maximum horizontal load and using NLMR4. Softening tensile behavior is represented by an "x", crushing in compression by a "c".

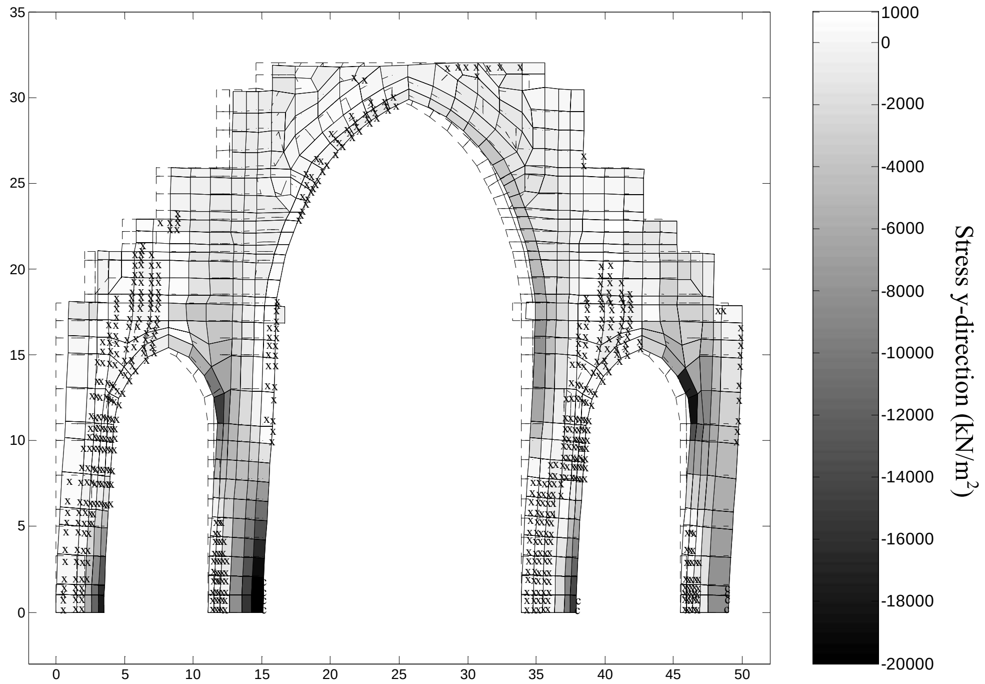


Figure 6 : Stress σy for the maximum horizontal load (NLMR4)

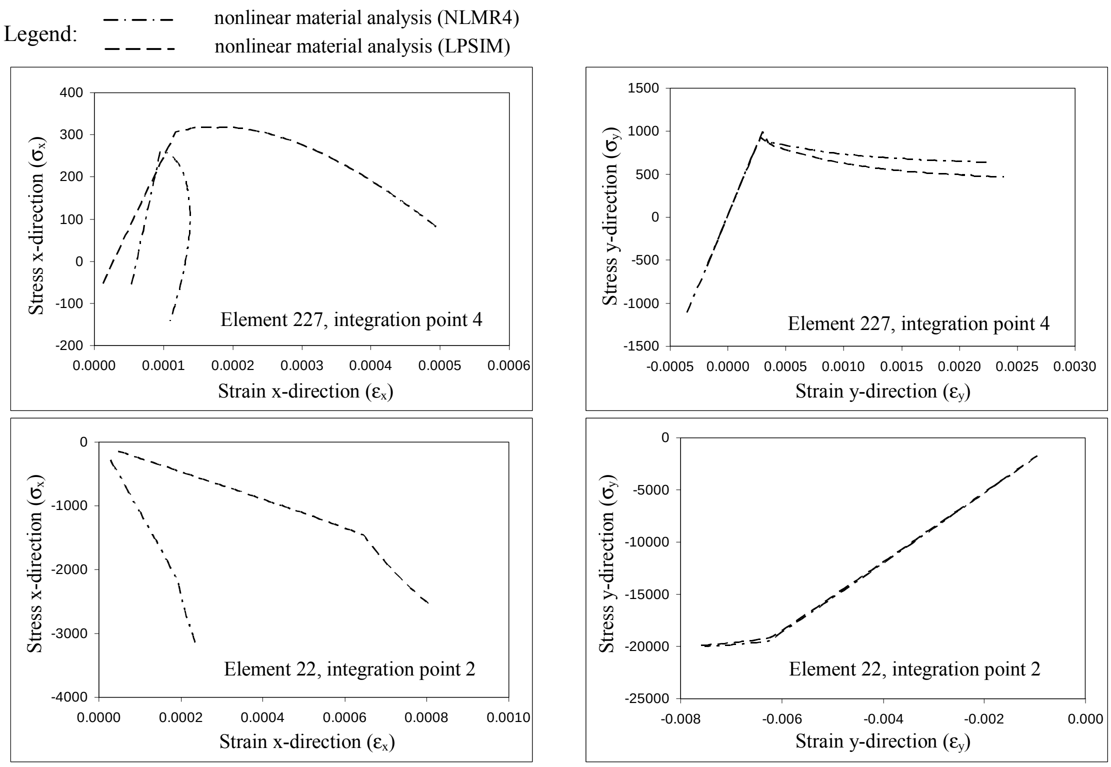


Figure 7 : Comparison of Stress vs. Strain relationships

Stresses vs. Strains relationships are shown in Fig. 7. Location of the elements is given in Fig. 4. Numbering of integration points is counter-clockwise and starting by the lower-left point of the membrane element (2x2 Gaussian points). In element 227, the two tensile softening behaviors in the vertical direction agree. Horizontal strains are larger with LPSIM. Analogously, in element 22 this phenomenon exists, but softening behavior occurs in compression stage. In both cases, NLMR4 reduces strains.

4 CONCLUSIONS

Three formulations of membrane elements have been presented. All formulations have been validated through simple tests. In the linear elastic analysis of one cross section of The Fatih Mosque, the vertical displacement component at joint 737 obtained with LMR4 duplicates that obtained with LPSIM while the horizontal component agree. Drilling DOF's give flexibility to the linear membrane formulation. In these two formulations, incompatible displacement modes are added.

In the material nonlinear analysis, NLMR4 is stiffer than LPSIM. It shows that the linear plane stress formulation containing incompatible displacement modes is more flexible than the nonlinear membrane formulation with drilling DOF's and no incompatible displacement modes. It seems to be that incompatible displacement modes need to be included into the nonlinear membrane formulation as well.

ACKNOWLEDGEMENTS

This study has been supported by The Scientific and Technological Research Council of Turkey (TUBITAK) under Project no. 105T047.

REFERENCES

Geyer S. and Groenwold A.A. 2002. Two hybrid stress membrane finite element families with drilling rotations. *International Journal for Numerical Methods in Engineering* 53, p. 583-601.

Ibrahimbegović A. 1993. Mixed finite element with drilling rotations for plane problems in finite elasticity. *Computer Methods in Applied Mechanics and Engineering* 107, p. 225-238.

Ibrahimbegović A. and Frey F. 1992. Membrane quadrilateral finite elements with rotational degrees of freedom. *Engineering fracture Mechanics* 43, p. 13-24.

Ibrahimbegović A. and Frey F. 1993a. Finite element analysis of linear and non-linear planar deformations of elastic initially curved beams. *Int. J. for Numerical Methods in Engineering* 36, p. 3239-3258.

Ibrahimbegović A. and Frey F. 1993b. Geometrically non-linear method of incompatible modes in application to finite elasticity with independent rotations. *Int. J. for Num. Meth. in Eng.* 36, p. 4185-4200.

Ibrahimbegović A. and Frey F. 1995. Variational principles and membrane finite elements with drilling rotations for geometrically non-linear elasticity. *International Journal for Numerical Methods in Engineering* 38, p. 1885-1900.

Ibrahimbegović A. and Wilson E. L. 1991. A unified formulation for triangular and quadrilateral flat shell finite elements with six nodal degrees of freedom. *Commun. in Applied Num. Meth.* 7, p. 1-9.

Ibrahimbegović A., Taylor R. L. and Wilson E. L. et al. 1990. A robust quadrilateral membrane finite element with drilling degreees of freedom. *International Journal for Numerical Methods in Engineering* 30, p. 445-457.

Lourenço, P.B. 2002. Computations on historic masonry structures. *Progress in Structural Engineering and Materials* 4, p. 301-319.

Lourenço P. B., De Borst R. and Rots J. G. 1997. A plane stress softening plasticity model for orthotropic materials. *International Journal for Numerical Methods in Engineering* 40, p. 4033-4057.

Mena A. 2006. Development of 2D Nonlinear Finite Element Algorithms and Computational Procedures for Historical Structures. MSc. Thesis, Bogaziçi University, Istanbul, Turkey.

Mena A. and Fahjan Y. M. 2005. Practical aspects in modeling and nonlinear analysis of historical structures. *Proceeding of The International Earthquake Engineering Conference (TINEE), November 21-24, Dead Sea, Jordan*, paper no. 19.

Wilson E. L. and Ibrahimbegović A. 1990. Use of incompatible displacement modes for the calculation of element stiffnesses and stresses. *Finite Elements in Analysis and Design* 7, p. 229-241.

Zhu, Y. and Zacharia T. 1996. A new one-point quadrature shell element with drilling degrees of freedom. *Computer Methods in Applied Mechanics and Engineering* 136, p. 165-203.