Limit Analysis of Masonry Constructions by 3D Funicular Modelling

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ABSTRACT: A method is presented for the analysis of complex masonry constructions by the computer simulation of networks of strings describing funicular states of equilibrium. The method can be applied to 3D framed structures and shells (including vaults and domes). Thanks to the adoption of an efficient and accurate cable element, the method can deal with complex and multiple hanging nets composed of numerous virtual strings subjected to arbitrary loading. The formulation requires solving large systems of non-linear equations with iterative numerical procedures. Different optimization algorithms have been implemented to ensure the robustness of the method at solving the resulting highly non-linear equations. Complementary algorithms are implemented allowing the application of the limit theorems of plasticity. The application of the technique to the assessment of a masonry construction (the towers of the façade of Barcelona Cathedral) is presented in order to illustrate its applicability to the study of real constructions.

1 INTRODUCTION

The catenary is defined as the shape of a hanging flexible chain or cable when supported at its ends and acted upon by a uniform gravitational force. The shape of a stretched string under a given set of loads is also known as the funicular curve, from the Latin for string. The same term (or sometimes anti-funicular) is applied to arch profiles designed to resist the applied loads in pure compression.

As is well known, it was Robert Hooke who discovered that the ideal shape of a masonry arch in equilibrium with certain loads was that of the inverted catenary curve drawn by a chain subjected to the same weights. He apparently announced that he had made the discovery to the Royal Society of London around 1671, but he did not provide any details until 1675, and even then the details were encrypted. At the end of a printed lecture on Helioscopes and some other instruments in 1676 he inserted the following problem: “The true mathematical and mechanical form of all manner of arches for building, with the true butment necessary to each of them. A problem which no architectonick writer hath ever yet attempted, much less performed”. He then provided the solution in the form of an anagram whose decipherment was only revealed after his death in 1705. The solution read: "Ut pendet continuum flexile, sic stabit contiguum rigidum inversum” -as hangs a flexible cable, so inverted, stand the touching pieces of an arch (Heyman 1989).

The equation for the curve of a hanging flexible line or catenary was derived by David Gregory, who by 1698 had independently reached and expanded Hooke's assertion: “When an arch or any other figure is supported, it is because in its thickness some catenaria is included”.

The possibility of analysing masonry arches by the analogy between the equilibrium of compressed members with that of funicular models working in tension was used throughout the 18th and the 19th, and even at the beginning of 20th c., for the design and assessment of bridges or...
masonry structures composed of arches. A well known example is the study of the dome of St. Peter in the Vatican by Poleni (1743). At the beginning of 20th c., architect A. Gaudí used the same principle to design complex structures, such as the Church of Colònia Güell near Barcelona, by means of 3D hanging models (Fig 1, left).

Later in France, during the 18th c., La Hire, Coulomb and Coulomb undertook the problem from a different approach. Their understanding stemmed from regarding the arch as a conjunction of rigid bodies which could experience relative live displacements. According to Couplet, the collapse occurs when the arch develops a sufficient number of hinges (or sections experiencing a relative rotation) to become a mechanism (Heyman 1976). The first general analysis of the mechanics of arches was published by Coulomb in his 1773 essay. Coulomb developed a consistent and general theory of stability of arches which provided the mathematical base for the description of the different possible modes of collapse, including the effect of both relative rotations and sliding between parts. He also stated that the failure due to sliding is rare and suggested to consider only overturning (rotational) failures for practical purposes. He suggested the use of the theory of maxima and minima to determine the position of the more unfavourable hinges or sections of rupture (Huerta 1996).

A further development arrived with the thrust line theory and graphic statics during the 19th c. Graphic statics provided a consistent and practical method for the analysis of masonry structures and was actually used for the assessment of a large amount of masonry bridges and large buildings. An example is given by Rubio’s analysis of the structure of Mallorca Cathedral (Rubió 1912, Fig. 1, right).

Heyman’s (1966) formulation for the plastic (or limit) analysis of masonry arches synthesizes all the mentioned historical approaches and provides a theoretical and comprehensive base for the analysis of this type of constructions. According to this formulation, the limit theorems of plasticity can be applied to masonry structures provided that they verify the following conditions: (1) The compression strength of the material is infinite; (2) Sliding between parts is impossible; (3) The tensile strength of masonry is null. These conditions enable the application of the well known limit theorems of plasticity.

In particular, these conditions allow the following formulation of the lower-bound (or safe) theorem: The structure is safe if a thrust line can be found, in equilibrium with the external loads, within the boundaries of the structure. A corollary of the safe theorem, the so-called uniqueness theorem, is also applicable: The collapsing condition of the arch will be reached if a
thrust line can be found for which there are as many points tangential to the boundaries of the structure as hinges necessary to convert the structure to a mechanism. When this occurs, the sections where the thrust line is tangent to the boundary become the location of actual hinges, the determined load is the true ultimate load and the mechanism is the true ultimate mechanism.

More recently, the use of the aforementioned principles in combination with modern computers and advanced numerical methods has provided powerful tools for the analysis of masonry constructions. Livesley, (1978), Gilbert and Melbourne (1994), Baggio and Torvalusci (1998), Orduña and Lourenço (2003), among others, have proposed different methods for the assessment of masonry structures by limit analysis.

Most of modern computer developments based on limit analysis have exploited the possibilities of the upper-bound theorem (cinematic approach) through the consideration of a rigid block scheme. A lesser number of numerical proposals explore the potential of the static approach. A formulation for the analysis of curved shell masonry members has been presented by O’Dwyer (1999), consisting of the decomposition of the shell element into a system of arches in equilibrium; by applying the safe theorem and maximizing the ultimate load, via an optimization process, O’Dwyer establishes a procedure which allows the general application of Heyman’s limit analysis to vaults and domes.

The method presented here is based on the lower-bound theorem (static approach) and follows the wake of Hooke’s and Gregory’s equilibrium curves or Gaudi’s funicular nets. The method consists of a computer technique for the assessment of complex masonry constructions, including 3D framed structures and shells; skeletal masonry constructions are modelled as 3D catenary nets composed of numerous virtual strings subjected to arbitrary loading. The formulation requires solving a large system of non-linear equations by iterative numerical procedures. Different optimization algorithms have been implemented to ensure the robustness of the method.

2 ANALYSIS OF 3D MASONRY SKELETAL STRUCTURES BY FUNICULAR NETS

2.1 Method of analysis

The method proposed is based on the description of the equilibrium lines of 3D skeletal masonry structures by means of virtual funicular (or cable net) systems. The masonry structure is assessed by the application of the safe and uniqueness theorems. The method includes the following features: (1) A numerical catenary element used to describe the geometry and equilibrium condition of each different individual cable or segment of a cable; (2) a criterion for the assemblage of the entire net of cables and the derivation of the global system of equations; (3) an efficient strategy for the resolution of the resulting system of highly non-linear equations; (4) a method for the description of the strength boundaries (external surfaces) of the structural members composing of the real masonry construction; and (5) specific algorithms for the determination of optimal solutions complying with the limit theorems of plasticity. In particular, the safe and uniqueness theorems can be used to assess the safety and to predict the ultimate mechanism of the structural systems. Some of these features are described below in more detail.

The validation of the method by comparison with simple cases with known experimental and analytical results has been already presented (Andreu et al. 2004). In the present paper, the application to the analysis of a 3D, more sophisticated construction—the lateral towers of the façade of Barcelona Cathedral, built at the end of 19th c.—is presented to illustrate the capacity of the method to deal with real and complex structures.

2.2 Adopted catenary element

Particular attention has been paid to the selection of a type of cable element adequate for the modelling of the equilibrium lines describing the equilibrium of the masonry structure. The possibility of using an inextensible cable element has been disregarded because of the severe numerical problems (such as numerical instability) experienced by computer applications which handle this type of element. Instead, a modified elastic catenary element has been adopted for this purpose. The elastic catenary element is defined as the curve adopted by a rigid elastic cable with negligible flexural stiffness, with fixed ends and submitted to the effect of gravity. A dis-
cussion on elastic catenaries can be found in Irvin e (1981) and Tibert (1999). It should be noted that the conventional formulations are based on the hypothesis of small deformations, meaning that the forces are integrated with respect to the initial configuration of the catenary. Hence, the weight per unit length does not vary consistently with the elongation experienced by the catenary. However, in a real elastic cable, the weight per unit length will actually decrease in value as the cable elongates. This may result in an inaccurate equilibrium of forces in the deformed configuration. For the present method, a modified formulation of an elastic cable element has been derived which, while preserving the condition on small deformations, ensures the conservation of the unit weight of the cable after the elongation (or, equivalently, the conservation of the mass of the cable after elongation).

The solution of problems involving a net of cables requires the formulation of an elementary stiffness matrix (Tibert 1999). In the numerical application, the stiffness matrix of the cable elements is constructed by using an exact analytical expression previously derived. An analytical construction of the stiffness matrices is preferred to a possible numerical estimation (using, for instance, finite differences) because it provides larger efficiency and numerical stability. The advantage of using the analytical expression is particularly remarkable in the case of complex cable networks involving thousands of elements. The formulation of the cable is given in Andreu et al. (2004) and Andreu (2006).

The equations of the different cables composing the net can be assembled by imposing the conditions of equilibrium and kinematic compatibility at each node. For that purpose, conventional procedures such as those used in matrix calculation for frame structures can be considered. The resulting non-linear system that arises in each load increment is solved iteratively by means of the Newton-Raphson method. This procedure enables the efficient solving of complex systems including hundreds or thousands of cable elements (Fig. 2).

![Figure 2: Examples of funicular nets created and solved by the proposed method](image)

### 2.3 Application of the limit theorems of plasticity

The geometric safety factor introduced by Heyman (1982) is considered here to numerically apply the safe and uniqueness theorems. For a given loading, the geometric coefficient is obtained by comparing the depth of the real arch with that of a minimum arch for which equilibrium is still possible (i.e. for which it is still possible to fit a thrust line). A solution complying with the safe theorem is obtained by an optimisation procedure involving the maximisation of the geometric safety factor. The lengths of the elements ($l_k$) which compose the mesh and the coordinates ($x_i$) of the degrees of freedom constrained by external supports are taken as the variables of the optimisation problem. The vector of the variables of the problem is thus defined as $X = \{x_1, ..., x_n, l_1, ..., l_m\}$, $n$ being the number of cable elements and $m$ being the number of nodes constrained. The target function should evaluate the minimum distance of any part of the fu-
nicular network to the boundaries of the structure. The process of optimization requires the maximization of the mentioned minimum distance.

The target function is defined in Fig. 3, where $D_{ij}$ is the distance between node $i$ of the cable network and node $j$ of the external surface, $m_{res}$ is the number of nodes of the external surface and $k=1,2,...,m_{ext}$ is the subset of nodes of the cable net which lay beyond the resisting surface. The target function will evaluate in a general manner the minimum distance of the funicular network to the boundary of the masonry structure and has positive value if the entire network is situated inside the boundary and negative value if part of it falls outside the boundary (Fig. 3, b). This construction is needed to ensure that function $D_{min}$ provides a meaningful measure of the geometric safety coefficient.

A complementary optimisation algorithm has been introduced to determine the ultimate load (and collapsing mechanism) through the application of the uniqueness theorem. In this case, the output of the process is the maximum load multiplier for which the maximum possible value of $D_{min}(X)$ is still positive. Two optimisation methods have been implemented: the steepest descent method (gradient method) and the Fletcher-Reeves method (Vanderpaats, 1984). Both have shown to produce robust procedures even for problems involving a very large number of cables.

3 EXAMPLE OF APPLICATION. TOWER OF FAÇADE OF BARCELONA CATHEDRAL

The possibilities of the method are shown through its application to the study of one of the lateral towers of the façade of Barcelona Cathedral (Fig. 4). The study is part of the investigations carried out for the restoration of the façade and cimborio, both built in modern times over the existing, medieval structure (the façade and its towers at the end of 19th c., the spire of the cimborio at the beginning of 20th c.). The towers are modest buildings rising 24 m over a square base 5.3 m wide. Their upper volume consists of a octagonal pyramidal spire 10 m high (Fig. 4b) crowned by a weighty pinnacle shaping a 3D cross. The towers constitute a new structure built over medieval walls and vaults at a height of 26 m over the ground level of the Cathedral. The structure is made of regular sandstone masonry with average compression strength over 50 N/mm$^2$. The construction includes a set of original iron rings embedded in the masonry within each horizontal nerve of the spire (Fig. 4c) and also at level of the springing of the vaults of the inferior body and at the base of the vertical nerves of the spire. Using the method herein presented, the structure of the towers has been assessed for both gravity and wind loading. The application of the method requires, as a previous step, the full definition of the limiting strength surfaces and the initial un-deformed cable model (Fig. 5a-c). In a first attempt, the possible contribution of the iron rings was ignored in the calculations.

Gravity loads are applied on the strings and nodes of the model according to their real distribution in the structure. For gravity loading, applying the method leads to the determination of a possible deformed configuration of the funicular model in equilibrium with the existing weight. As mentioned, the application of the safe theorem, following the process described in section 2.3, leads to the determination the solution with the maximum geometric safety factor, i.e. with maximum overall distance between cables and strength surfaces. As shown in Fig. 5d-e, the so-
The solution obtained is totally contained within the strength boundaries, meaning that the structure can resist safely the entire dead load even if no action of the iron rings is considered. The thrust lines reach a minimum distance to the strength boundaries (about 6 cm only) at the base of the spire.

Figure 4: (a) Façade of Barcelona Cathedral c. 1900 with its lateral towers already built; (b) View from the interior of the spire of the towers; (c) detail showing a reinforcing iron ring, after the removal of its stone cover, placed between two tracery panels.

Figure 5: (a) Modelling of the strength surfaces; (b) and (c) initial funicular model (reversed); (d) deformed funicular model; (e) detail of the spire in the deformed funicular model.

The study also illustrates that the transverse elements of the spire, consisting of monolithic frames of stone tracery, are also needed to reach an adequate state of equilibrium in dead loading; such elements contribute with a radial outward horizontal thrust (an inward one in the unreversed model) which deviates the vertical cables and prevents them from exceeding the strength surface.
Additional calculations are made by considering a certain balancing action of the iron rings. For that purpose, the rings were modelled as weightless straight elements experiencing tension in the un-reversed funicular model (and compression in the reversed, anti-funicular one). In that case, the solutions obtained are less strict and show a significant increase of the distance between cables and strength surfaces in the critical regions.

Wind analysis has been carried out in a simplified way by applying a set of static forces representing wind pressure. A wind speed of 160 km/h, multiplied by a safety factor of 1.5, has been considered following the requirements of the Spanish building code for the location and exposure of the building.

As can be observed in Fig. 6a-c, the funicular model, once equilibrated and optimized, deviates very significantly from the volume of the structure, meaning either that the structure can not resist the wind forces or that the true response of the structure is not adequately described by the simple funicular scheme considered. In fact, the model is not considering the contribution to the overall strength of the iron rings and the tracery monolithic panels which close the spire. Thanks to the tensile strength of stone (which is normally not considered in the calculations) the monolithic tracery frames can work, to a certain extent, as stiff frames. The joint contribution of the tracery panes and the confining iron rings has been modelled in an approximate way by including a set of stiff, weightless struts, working either in tension or in compression in the model (Fig. 6d). In that case, a more revealing solution is obtained which remains adequately contained within the strength boundaries of the structure.

![Figure 6: Funicular solutions for wind loading without (a-c) and with stiff braces (d) simulating the stiffening action of ties and monolithic tracery panels](image)

4 CONCLUSIONS

A computer method for the analysis of 3D masonry structures using the antifunicular principle (the analogy between the equilibrium of arches with that of hanging funicular systems) has been presented. Auxiliary stiff elements can be included in the funicular nets to model strengthening devices such as stone or wooden struts, or iron ties and rings.

The technique includes specific optimization algorithms for the application of the safe (or lower-bound) and uniqueness theorems to assess the safety condition of the structure. The ap-
Application of the technique to the analysis of a complex structure has shown its applicability and possibilities.

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