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Circular, Pointed and Basket-Handle Arches: A Comparison of Structural Behavior of Masonry Spans

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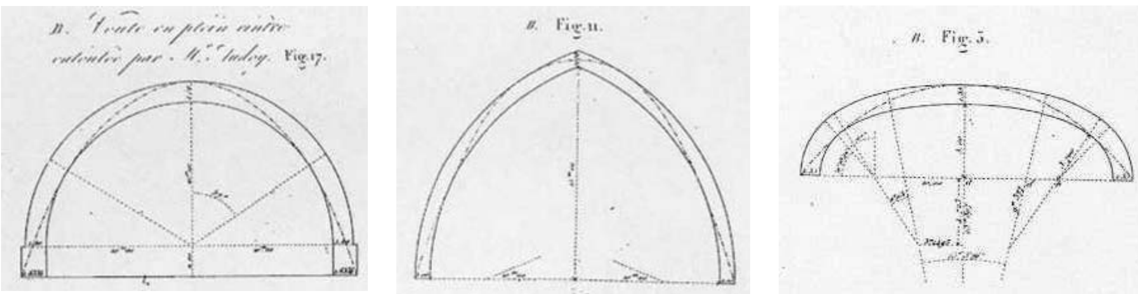
ABSTRACT: This paper provides new insights into the relationship between the geometry and structural performance of masonry arches. Comparing step by step the behavior of circular, pointed and basket handle arches, new topics like maximum and minimum thrust and allowable displacements that can be tolerated by the structure in equilibrium have been investigated in detail. Traditional limit analysis, with the hypotheses of zero tensile strength, infinite compression strength and no sliding, has been used for the evaluation of these characteristic values.

1 INTRODUCTION

Although masonry arches are one of the most used structures in ancient constructions, a quantitative static theory for arches was not defined until the end of XVII century with De la Hire (1731), Belidor (1737) and Couplet (1729) who modeled the construction as rigid blocks. Geometric rules for abutment proportions can be found in Vitruvio (I b.C.), Leonardo da Vinci (1505) and later in Boccojani (1546), Martìnez de Aranda (1590) and Derand (1643). Huerta (2004) has demonstrated that proportional design of masonry structures was used by master builders up to the 19th century.

A comprehensive study of different arches shapes was conducted by Méry in 1840 (Fig. 1). He analyzed the different position of the hinges and the subsequently failure mode through the thrust line, a theoretical path of the resultants of the compressive forces through the stone structure. The concept of the thrust line was framed in the wider theory of limit analysis developed for masonry structures by Heyman (1966).



(a) (b) (c)

Figure 1 : Méry (1840) demonstrated thrust lines for arches of various shapes

2 PROBLEM FORMULATION

Over the centuries, researchers have focused mostly on circular arch geometry. Pointed arches – adopted in Gothic cathedrals – and basket handle arches – mainly used for bridges – have also been widely used in historical constructions. Until now, the structural behavior of pointed and basket handle have not been researched in sufficient detail. In the relevant literature there are no expressions or indications of the values of maximum and minimum thrust. In addition, the allowable displacements that can be tolerated by the structure in equilibrium have not been investigated.

3 METHODOLOGY

Through the application of equilibrium conditions, a rigorous numerical algorithm has been developed and implemented in the commercially available software MATLAB. The procedure calculates the minimum possible thickness of arches, the range of thrust values, the allowable displacements of one support and the variation of thrust due to the span increase. These parameters have been obtained through the construction of the thrust line for different geometries. These values can be found by examining the equilibrium of the central region of an arch. For example, to determine the minimum thrust, the correct hinge location is positioned where the thrust from the central region is maximized (Coulomb 1773; Heyman 1972). To calculate the maximum allowable spreading of the support of an arch, the approach developed by Ochsendorf (2002, 2005) was followed.

In order to have a visual check of the results gained through the numerical code, interactive graphic static analysis was implemented in the drawing software CABRI following the methods developed by Block (2005). The graphic static method is as rigorous as the numerical one based on the equilibrium of the voussoirs. The shape of the thrust line in an arch can be found through graphical methods by using the funicular polygon as detailed by Zalewski and Allen (1998) and others. For any arch there is a family of possible lines of thrust and the arch is generally statically indeterminate. However, support movements can lead to the creation of hinges, and the thrust line in a three hinge arch can be found uniquely. Interactive graphic statics makes it fast and easy to verify the accuracy of all analyses. Examples can be seen at the web page developed at MIT by Philippe Block at http://web.mit.edu/masonry/interactiveThrust. By using such interactive drawing programs, parametric models are created which provide quick tools to compare the relative influence of geometry on the range of possible stable conditions.

The numerical and graphical analyses have been verified against each other and later compared to the results gained by experimental tests by the authors at MIT on circular and pointed model arches.

The masonry arches considered here have the following characteristics. The circular arch has only one center of curvature in the middle of the span; the pointed arch has one center of curvature on each side of the vertical axis of symmetry; and finally, the basket handle arch has three centers of curvature: the two external ones placed on each side of the vertical axis of symmetry and the third one on the vertical symmetry axis so that the intrados of the arch has a smooth curve.

In order to assess their structural behavior, a parametric analysis has been performed by varying some fundamental ratios keeping constant the angle of embrace of 90°. In this paper, the circular arch has been considered as the "reference arch" for the pointed and the basket handle arch because of its better known behavior. Some fundamental ratios have been defined for the complete definition of the geometry. The value of the centerline radius of the circular arch R_{circ} over the arch thickness t is the common parameter for the three types of arches. In addition to R_{circ}/t, in pointed arches the ratio e/R_{circ} between the eccentricity e of the centers of the arches normalized to R_{circ} has been considered. For basket handle arches the ratio b/R_{circ} of the smaller radius b over R_{circ} and the ratio d/R_{circ} of the distance of the middle center d from the impost of the arch and again R_{circ} have been considered (Fig. 2).

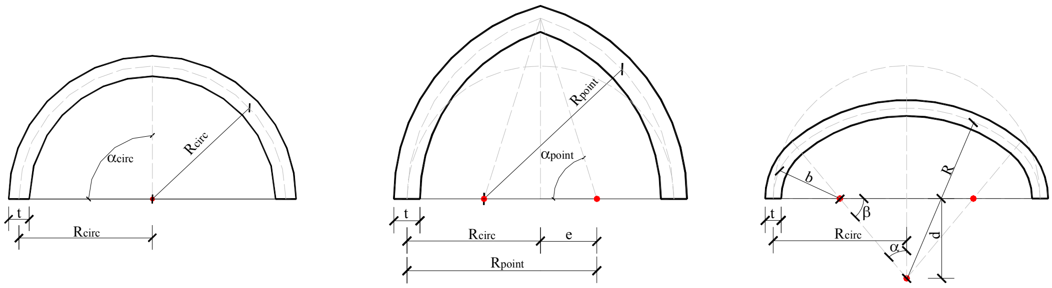
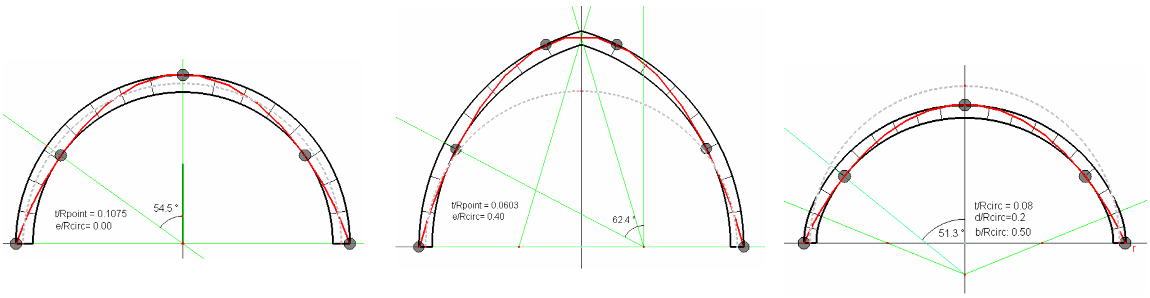


Figure 2 : Geometry definition for the three types of arches.

4 MINIMUM THICKNESS

According to limit analysis, a real arch has a thickness sufficient to accommodate infinite possible thrust lines but there will only be one depth for which the arch is on the point of collapse without any movements of the supports, and that is the minimum possible thickness (Heyman 1969). A thinner arch can not be constructed without the line of thrust passing outside the masonry, which would imply tension in the material in contradiction to the no-tension assumption. It also implies that for this thin arch the minimum and the maximum thrust will coincide. As shown in Fig. 3, the circular arch is on the point of collapse by the theoretical formation of a five-bar chain, the pointed arch will have one additional hinge (that is, a six-bar chain) that will disappear for any slight asymmetry and the basket handle arch is again a five-bar chain. In all the cases the mechanism will be reduced to the regular four-bar chain because of the suppression of one of the abutment hinges.



(a) (b) (c)

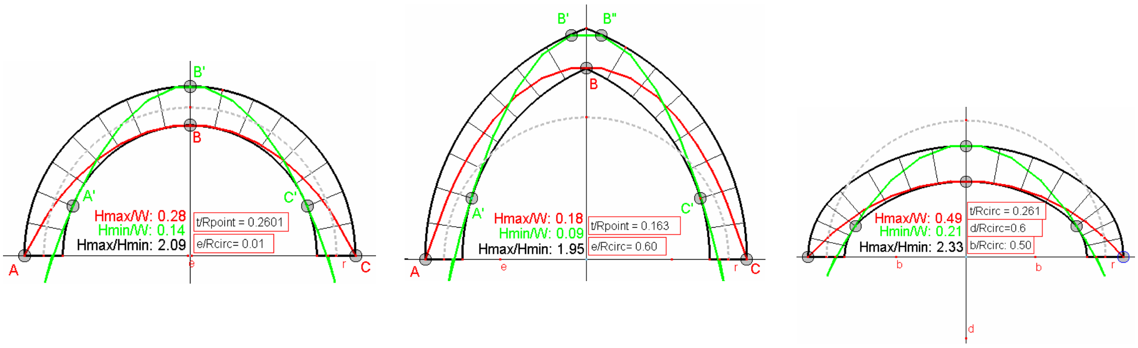
Figure 3 : Minimum thickness: (a) circular arch; (b) pointed arch; (c) basket handle arch.

5 THRUST VALUES

According to the "safe theorem" of limit analysis, if it is possible to find at least one thrust line entirely contained in the shape of the arch, then the arch is "safe" (Heyman 1995). Generally, as soon as a thicker arch than the minimum one is considered, some thrust lines lie in the depth of the structure. Defining the minimum and the maximum thrust as the limits of the range of variability, a cracked arch will develop a minimum thrust when the supports are moved apart and a maximum one when the supports are moved together.

Fig. 4 shows these two ideal lines in the three arches allowing some remarks. The maximum thrust line may touch the boundary in three points so that only three hinges need to form in whatever arch. The minimum thrust line will again develop three hinges in the circular or basket handle arch, but four hinges must form in the pointed arch because of the impossibility of the thrust line to be pointed. But any slight asymmetry, whether of geometry or loading of the arch, will ensure that only one of the hinges near to the crown will form, so that again a stable three-hinge mechanism will form. Therefore, the two hinges at the crown function as one hinge.

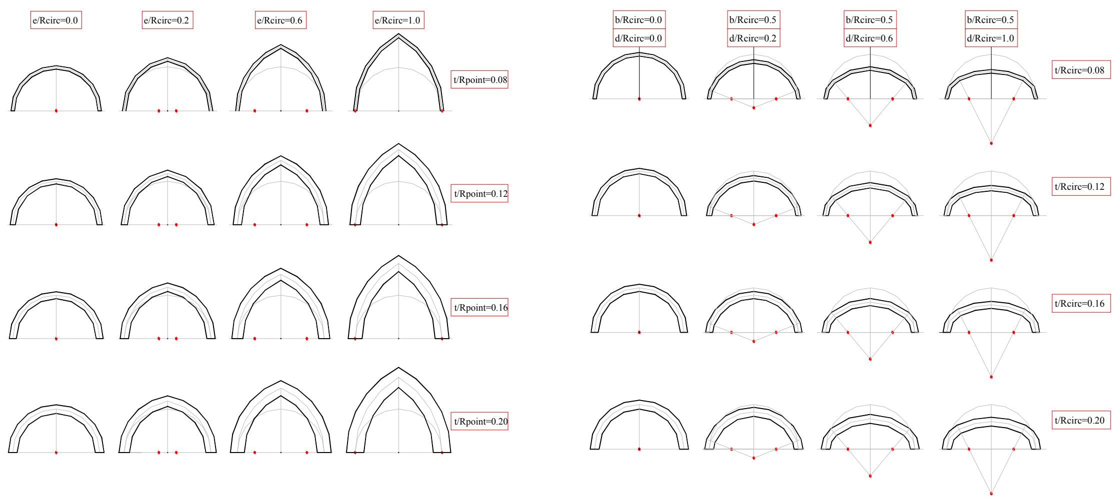
Furthermore in Fig. 4, the three structures have been depicted so that the span and the thickness of the voussoirs are the same. As expected, the pointed arch assumes lower values of maximum, minimum thrust and thrust ratio compared to the circular arch. On the contrary, the basket handle arch develops larger thrust values both in terms of maximum and minimum as well in terms of their ratio.



(a) (b) (c)

Figure 4 : Thrust lines: (a) circular arch; (b) pointed arch; (c) basket handle arch.

As before, a systematic analysis on these structures with the angle of embrace of 90°, varying the ratios e/R_{circ} (0, 0.2, 0.6 and 1), and t/R_{point} (from 0.04 to 0.24 stepwise 0.04) for pointed arches and b/R_{circ} (fixed at 0.5), d/R_{circ} (0, 0.2, 0.6 and 1) and t/R_{circ} (from 0.08 to 0.24 stepwise 0.04) for basket handle arches has been carried out. In Fig. 5 some samples of the considered arches are illustrated.

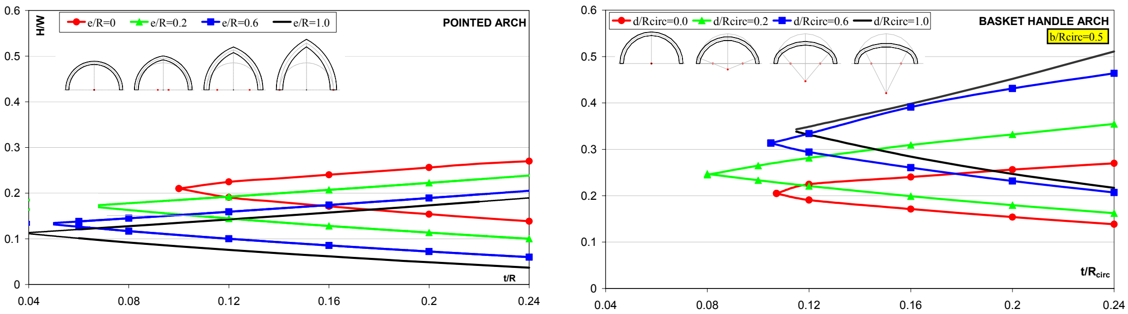


(a) (b)

Figure 5 : Study cases varying the geometries: (a) pointed arches; (b) basket handle arches.

To determine the range of variability of the thrust in the depth of the arch, a suitable algorithm has been written by the authors applying some equilibrium conditions to part of the structure. By varying the position of the hinges through the arch and applying the safe theorem, the algorithm maximizes the value of the thrust. This happens for both the search of the minimum and the maximum values, simply inverting the position of the hinges at the extrados or at the intrados of the structure.

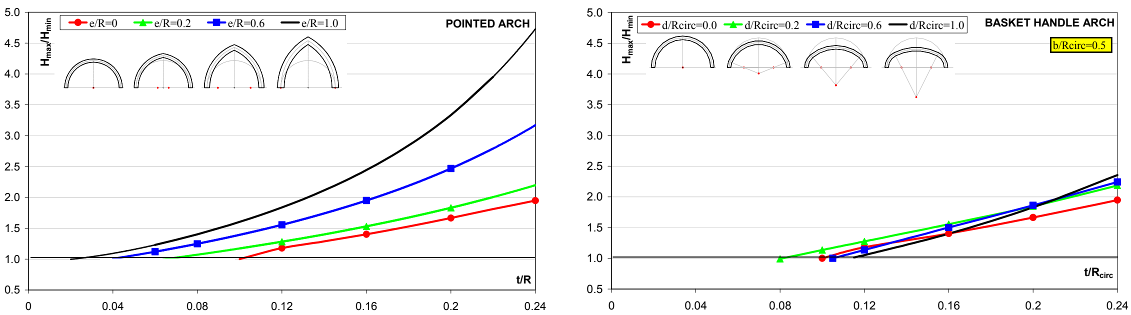
Because the geometry of the structure limits the range of possible thrust lines, the minimum and maximum thrust represent the smallest and the largest horizontal thrusts the structure transfers to its supports. In Fig. 6 these values, normalized to the weight of the arch (H/W), are depicted as a function of t/R for pointed and basket handle arches. In particular, in the first plot e/R_{circ} (from null eccentricity -circular arch- to the maximum one) and in the second one d/R_{circ} (from null arm -circular arch- to the maximum one) is varied. “Scissors shape” curves, representing the values of the minimum and maximum thrust are drawn; obviously, there will be a unique value of this ratio for which the maximum and the minimum thrust coincide and this is exactly the value of the minimum possible thickness of the arch for the chosen geometry. Some remarks can be made for both of the plots. In the first diagram, representative of the pointed arches, it can be seen that increasing the pointedness of the arch, smaller thickness can be obtained and the values of the two thrusts decrease as well. In the second graph, describing the behavior of basket handle arches, it is highlighted that three centers arches assume higher values of thrust than the circular ones. Surprisingly, the basket handle arch with a small distance from the impost (d/R_{circ}=0.2) can assume thinner values than the circular one.



(a) (b)

Figure 6 : Maximum and minimum thrust: (a) pointed arches; (b) basket handle arches.

A visualization of the ratio H_{max}/H_{min}, for both types of arches, can be seen in Fig. 7 where the minimum values are fixed to unity (i.e. the minimum thickness). It can be noticed that increasing the "pointedness", higher values of H_{max}/H_{min} and thinner arches can be obtained. The same distinct variability of values can not be detected for basket handle arches, where the results are close to each other for varying geometries.



(a) (b)

Figure 7 : Ratio of the maximum over the minimum thrust: (a) pointed arches; (b) basket handle arches.

6 HORIZONTAL SUPPORT DISPLACEMENT

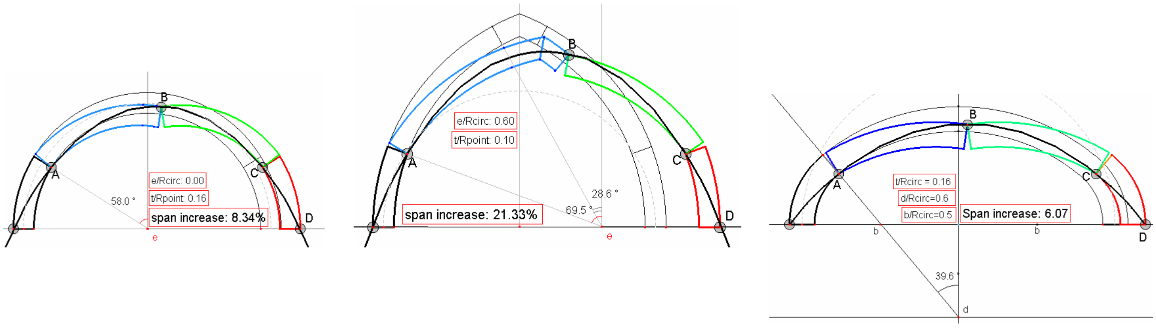
Historic masonry structures are commonly subjected to differential settlements in foundations, defects in constructions, consolidation of materials (creep in the mortar) and vibrations. The result of these actions can cause an increasing displacement over the life in the upper part of the constructions and the structure reacts by developing cracks, often showing large deformation capacity, for example in the case of leaning walls or buttresses. Arches on spreading supports could collapse in one of two ways: a five-hinge mechanism (symmetrical in the circular and in the basket handle arches and asymmetrical for the pointed one with the activation of only one of the two hinges close to the crown) or a three-hinge mechanism by snap-through if the thickness is sufficiently large. Generally, in the five-hinge collapse mechanism, the central portion of the arch is a three-hinged arch, which deforms to accommodate the span increase. Previous studies by Ochsendorf (2006) show that this result is unsafe since the hinge locations are not fixed, but may move during the support displacements toward the crown of the arch.

Neglecting the cause of the displacements, the stability of the structure has been investigated, through kinematic and static analyses. Through the use of two computer codes, Matlab and Cabri, the analysis of the evolution of the thrust line in the arch as the supports are spreading has been performed. Although the procedures are different, the results agree with high accuracy confirming the correctness of the two techniques. Following the method proposed and investigated by Ochsendorf (2002) for the circular arch, a new script has been written in Matlab for the spreading of the support of the pointed and the basket handle arch.

The main steps are as follows:

1. Determine the hinges' position for the minimum thrust condition.
2. Impose the support movements, constantly updating the position of the thrust line.
3. Continue until collapse occurs due to one of two modes. Most commonly, the collapse condition is reached when the thrust line reaches the exterior at one of the two supports. This means that a fourth hinge will form and the structure becomes an unstable collapse mechanism.
4. If the first collapse condition does not occur, the second one could occur due to snap-through of the arch.

Fig. 8 presents three sketches of the position of the thrust line for the various arches at the point of collapse due to spreading supports.

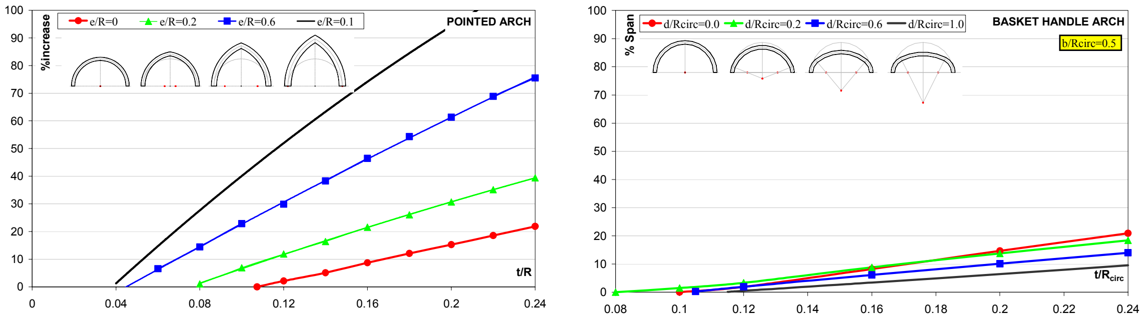


(a) (b) (c)

Figure 8 :Maximum displacements: (a) circular arch; (b) pointed arch; (c) basket handle arch.

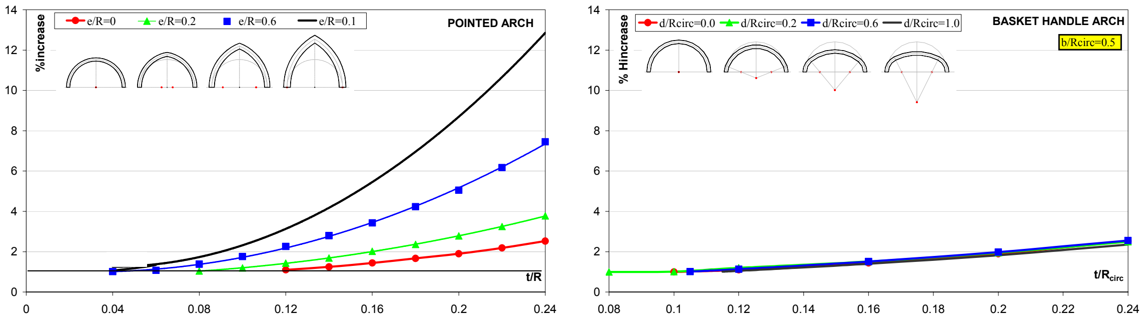
Values of the displacement due to the increasing of the span are presented in Fig. 9. It is clear that pointed arches are able to deform much more than the circular arches before the collapse. Also, basket handle arches are able to bear smaller displacements even than the circular arch.

In terms of thrust increase due to the span increase, the results for the two kinds of arches are depicted in Fig. 10. Again, a significant difference of behavior of pointed arches compared to basket handle arches can be highlighted. If in the first case the thrust increases when the pointedness changes, in the second one no major differences can be detected when the shallowness is increased.



(a) (b)

Figure 9 : Span increase in arches: (a) pointed arches; (b) basket handle arches.



(a) (b)

Figure 10 : Thrust increase due to the span increase: (a) pointed arches; (b) basket handle arches.

7 CONCLUSIONS

This paper has analyzed the structural behavior of masonry arches focusing on pointed and basket handle arches with a total embrace of 180°. The values of minimum thickness, minimum and maximum thrust, their thrust ratio, the maximum possible span increase and the change of the internal forces once one support is moved have been assessed analytically. All of the theoretical values have been checked against each other with two different codes, Matlab (analytical code) and Cabri (visual graphic code), and the results agree very well.

Pointed arches develop smaller values of thrust than the circular arch both in terms of minimum and maximum values. On the contrary, the ratio H_{max}/H_{min} assumes bigger values than the reference arch so that, more equilibrium configurations are possible. At the same time, thinner arches can stand once the pointedness is increased. Fairly different behaviors can be detected by arches shallower than the circular one. In basket handle arches the values of the maximum and the minimum thrust are increasing with the shallowness whilst their ratio is almost uniform. A surprising behavior of arches with a small distance from the impost of the arch has been assessed in terms of minimum thickness because of the possibility of standing for smaller values.

The theory developed by Ochsendorf (2002) on circular arches for the displacement of the supports has been confirmed for pointed and basket handle arches. The main result is that pointed arches can bear greater displacement than circular arches whilst basket handle can move less. Naturally, the more the arch is pointed the more it is capable of deforming. On the other side, basket handle arches can bear even less displacements than the circular arch and the thrust variation is fairly uniform.

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