A benchmarking study of the analysis of non-reinforced structures applied to the structural behavior of domes

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ABSTRACT: This paper presents a benchmarking study of two methodologies used in the analysis of non-reinforced structures: numerical and graphical methods applied to the structural behavior of domes. The structural analysis was performed through the Finite Element Analysis software ABAQUS. The graphical methods are based on the theory of limit analysis and they were carried out through CAD software tools Pro-Engineer and AutoCAD. In order to test both methodologies, they were applied to the structural behavior of the Pantheon in Rome. The results obtained through the numerical methods were compared with Mark & Hutchinson’s results. The graphical analysis seeks to obtain and visualize the line of thrust and the Stability Factor. Results obtained were compared with Lancaster’s results. The aim of the paper is to validate and to compare both methods.

1 INTRODUCTION

The Graphical Method used to study the structural behavior of domes seeks to obtain and visualize the line of thrust. This is a theoretical line that represents the path of the resultants of the compressive forces throughout the structure. Heyman (Heyman 1998) has provided the theoretical basis for thrust line analysis and has illustrated its application to domed structures. Numerous studies (Cowan 1977, Wolfe 1921, O’Dwyer 1999, Livesley 1992, Oppenheim 1992, Como 1992, Boothby 2001) have been carried out for determining the thrust line. As early as 1748, Poleni (Heyman 1988) analyzed St. Peter’s dome in Rome using equilibrium methods to estimate the internal line of thrust. Ochsendorf (Ochsendorf 2006) developed new interactive thrust-line analysis tools using limit analysis to illustrate possible collapse modes and to allow users to clearly visualize the forces within the masonry. Although it is impossible to know the actual thrust line, since there are many solutions, it is possible to establish its value within certain limits and to obtain the maximum and minimum values according to the two extreme positions of the line thrust.

The Finite Element Method can identify local areas with of significant tension, possible cracking or distress under normal service loading conditions. With those methods we can better understand the history of the construction process on building deformation and stress.

The Pantheon of Rome was chosen as a site at which the methods were intended to be validated. Also known as The Rotunda, it is an emblematic building constructed two thousand years ago. Cracks in the dome appeared a short time after its construction. Mathematics, geometrical relationships and numerical proportions were the main characteristics of Roman construction (Huerta 2006). Roman knowledge was based on experience and critical observation of masonry building processes. Stability, equilibrium and safety were achieved through correct geometry and knowledge of the materials used.

The materials used in the construction, geometrical data, design and structure were carried out by Terenzio (Terenzio 1934), de Fine Licht (de Fine Licht 1968), MacDonald (MacDonald 1976, 1982), and Moore (Moore 1995). In 1986, Mark and Hutchinson (Mark & Hutchison 1986) used finite element analysis to calculate elastic stresses in the dome of the Pantheon. Lancaster, (Lancaster 2005) conducted an equilibrium analysis to answer various questions about the structure and design of the Pantheon.

This paper seeks to validate the values obtained, comparing them with aforementioned studies. Once the values were validated, additional studies were conducted to reproduce the different stages of its construction. The results are discussed as well as the advantages and disadvantages of both structural analysis methods.
and the benefit of utilizing each one under different circumstances.

2 SHORT DESCRIPTION OF THE PANTHEON

2.1 History
The first Pantheon was built by Agrippa in 25–27 BD. It suffered severe damage in 80 AD during the great fire of Rome. Although it was rebuilt by Emperor Domitian, it was struck by lighting in 110 AD, during the reign of Trajan. The present Pantheon was built for the third time under Emperor Hadrian between 118–125 AD. That date is known by stamps on the bricks.

2.2 Geometry and structure of the Rotunda
The Rotunda is a circular space covered by a concrete hemisphere. The internal height of the circular wall is equal to the radius of the sphere of the dome. The dome is held up by a 6-meter thick wall. The wall is made of concrete and is covered with about 60 cm of brick. The wall is not solid; it contains cavities, chambers and is open towards the inside with large exedras arranged on three levels (Lancaster 2005).

Inside the wall there is an archway of bricks, known as a relieving arch, to support the upper wall over the openings. The relieving arch is a semicircle of thin bricks standing radially on end extending into the concrete wall. This arch distributes upper loads to the piers during the long time it takes for the concrete to cure, but after curing, it becomes an integral part of the wall. This archway of bricks was only part of the wall and did not extend into the dome. The entire structure is resolved with arches incorporated in the wall. Externally, the lower part of the dome is hidden by the upper storey of the cylinder, while the visible part is raised in seven stepped rings. The first ring is 2.25 m thick. Six other exterior step-rings present a slope 1:10 inwards. The last ring starts at 47° from the axis.

2.3 Materials
The composition of The Rotunda has been documented by de Fine Licht and MacDonald. The rotunda rests on a ring foundation made of opus caementicium, 7.30 m wide and 4.50 m deep. The concrete is made of travertine fragments in layers in a mortar of lime and pozzolana. The unit weight of the mass of concrete is calculated to be about 2,200 kg/cm². The lower level (1,310 cm) consists of alternate layers of travertine fragments and fragments of tuff (the caementae) in a mortar of lime and pozzolana. The core is faced with a thickness of about 60 cm of brick, the unit weight being 1,750 kg/cm². The concrete of the upper level (890 cm) of the wall is lightened by changing from travertine plus-tuff to brick plus-tuff, alternating layers of pieces of tuff and broken tiles or bricks also in the same mortar. The uppermost level (840 cm) of the wall consists of concrete, predominantly of broken bricks in mortar. The stepped rings are composed of layers of brick fragments set in mortar. The unit weight of the mass of concrete is calculated to be about 1,600 kg/cm². The tufo giallo and lightweight volcanic scoria is used for the top of the dome, the unit weight being 1,350 kg/cm².

2.4 Cracking
The dome and walls are cracked. Terenzio, during his inspection of the Pantheon in 1930, documented cracking in the wall and the dome. The cracks that he documented continue up the dome to an average of about 57° above the springing, according to Mark and Hutchinson. Cowan theoretically placed this point at 37° degrees from the axis.

3 STRUCTURAL ANALYSIS METHODS: STRUCTURAL BEHAVIOR OF THE DOME

A dome is one of the most difficult architectural forms to construct. It is designed by rotating an arch 360° around its central vertical axis. It is a vault with double curvature, capable of covering a large space without any interior supports. They have great structural influence; the double curvature improves the behavior of the compression-curvature forces. The study of their structural behavior is based on knowledge of material capacities, structural properties and construction techniques. In ancient times, the construction was based on geometrical rules. The relation between geometry and structural behavior was decisive. The most important thing was to ensure the stability and safety of the structure.

The dome studied in this article is an unreinforced concrete structure. For the equilibrium analysis, it is assumed that it is a non-tension material, has virtually infinite compressive strength and sliding failure will not occur (Heyman 1995). The only load applied is its own weight. This study was carried out using two different methods: graphical and numerical.

3.1 Graphical method
In this article this is based on Wolfe’s method for dome analysis (Wolfe 1921). He developed a graphical method based on membrane theory. He analyzed the dome as a radial series of segments, subdivided into voussoirs. The weight of each voussoir increases from crown to base and rests on the center of gravity cut by the axis plane. The force polygon allows one to predict meridional and hoop forces. A recent
The Pantheon's dome is viewed as a net formed by meridians and parallels. Internal meridional forces increase from the crown to the base. All meridional forces are compressive forces. Internal hoop forces act in the latitudinal direction in a parallel ring. The hoop forces allowed ring by ring construction of the dome. For a dome of zero thickness, the membrane theory predicts that the hoop forces change at 51°49' from compressive forces to tensile forces (Heyman 1995). A typical failure consists of the formation of radial cracks along its meridians that divide the dome into segments, due to hoop forces.

3.3 Methods of comparison

In order to compare both methods, first were validated with the results obtained by different authors (Mark & Hutchinson 1986 and Lancaster 2005). Secondly, a relation between each was sought. The values found in the center of gravity are compared to in-points that present singularities.

4 MODELS STUDIED

4.1 Base Dome: Models 1 and 2 of the dome with different densities

The first two models analyzed are solely of the dome. Those models are used to validate the values obtained with the numerical method by comparing them with the results showed by Mark and Hutchinson (Mark & Hutchinson 1986). Subsequently, they are analyzed by the graphical method and an assessment of the results is obtained by comparing them to the data previously recorded.

Two models were used. Model 1 consisted of a hemisphere with an interior radius of 21.81 m and a thickness of 1.5 m. According to Mark, 2,200 kg/m³ were taken as the unit weight. It was not common among Romans to use 2,200 kg/m³ for vaults, but we use that unit weight to be able to compare the results obtained in this article with their results. Model 2 included a hemisphere with a unit weight of 1,350 kg/m³ in the dome. Both have been solved with graphical and numerical analyses.

It is assumed that the base of the dome is supported only against vertical loadings. There are no horizontal buttresses. That simplification allows us to analyze the dome using the shell theory. The results obtained make it possible to know the maximum value of circumferential tensile stress.

4.1.1 Study carried out with Abaqus

The values obtained through the graphical analysis have been taken at the center of gravity of the volume of the each voussoir. (Figure 2)

4.1.2 Study carried out with graphical analysis

Figure 3 shows the section of the dome used to obtain results in the gravity center of each voussoir (volume) and the force triangle corresponding to the centroids.
The vertical line is the sum of each weight. The horizontal line is an arbitrary one; the meridional forces are parallel to segments that link two adjoining centroids. The hoop forces are perpendicular at the meridians and pass by the final points of the meridional forces where they intersect with the horizontal line.

The value of the magnitude of the meridional and hoop forces are measured and divided by the contact area, yielding data that will be compared with previous results.

Table 1 shows the results obtained at the base of the dome with both methods and the values presented by Mark and Hutchinson. Mark does not give data about the mesh used, which could also affect the accuracy of the results and the values of the meridional forces.

Additional information that is important to know is to identify the point at which the internal hoop stress changes from compression to tensile. The graphical analysis shows that in voussoir number nine, where the center of gravity is placed at 52.5° from the axis of revolution, the hoop stress changes. The numerical analysis coincides at 50°. These results correspond approximately to the values expected for a thin hemispherical dome [51°49′].

The vertical reactions in the base of the dome vary from 8.5T – in the exterior of the dome – to 6.5T in the interior.

4.2 Models 3–6: Influence of the uppermost level and rings

The next four models deal with determining the influence in the structural behavior of different construction levels. First, the lower and upper level are added to the dome, secondly the uppermost and finally the stepped-rings.

In order to evaluate the influence of the uppermost level and the stepped-rings, four models were created. In Model 3, the pier has been added to Model 2. The pier has been reduced to 5.5 m and without taking the chambers into account. In Model 4, the uppermost level has been added to Model 3. Both are assumed to use lightweight concrete (Figure 2). Model 5 is a full model, with stepped rings added. The density of heavy concrete was taken at 2,200 kg/m³ for unit weight. Model 6 is similar to Model 5 with different densities, 1,350, 1,600 and 1,750 kg/m³, according to figure 2. Models 5 and 6 deal with how to determine the influence of the stepped-rings built over the dome.

4.2.1 Study carried out with Abaqus

Figure 4 shows the models studied with Abaqus. Model 3 assumes that the dome is embedded at the base. That supposition modifies the values obtained. The traction zone changes, the tensile stress reaches a value of 1.21 kg/cm² and is placed approximately 72° from...
the axis (Figure 9a). The base of the dome is compressed and the values of the hoop forces vary from
−0.1 kg/cm² to −0.83 kg/cm² at the exterior of the cupola. The maximum value −0.83 kg/cm² is taken outside of the cupola and coincides with the line of discontinuity.

In Model 4, the lower zone of the dome is embedded in the uppermost level of the wall. The values of the table are taken at the base and at three points: where the rings start, where they end and at an intermediate point on the interior surface of the dome. The traction zone changes and is placed on the exterior of the uppermost level, reaching a value of 0.6 kg/cm². In Figure 5b we notice a line of discontinuity where the dome tries to open itself. At that point the value reached is −0.1 kg/cm².

Models 5 and 6 show the influence of the rings over the dome. The values obtained at the base of the dome in model 6 hardly present differences from model 4. The hoop forces are about 0.25 kg/cm² and the meridional forces about −2.7 kg/cm².

4.2.2 Study carried out with graphical analysis
Models 4 and 6 were analyzed with the graphical analysis method. The profile of the dome has changed. In Model 4 the dome begins where the uppermost level ends (Figure 5a). The embrace angle is approximately 70°, taken from axis.

In Model 6 the weight of each stepped ring has been considered and the profile of the dome is taken, as shown in Figure 5b.

The values obtained with the Graphical Analysis method were reached with the steps indicated in 4.2 and are taken at the center of gravity of each voussoir.

4.3 Models 7–9: cracked model
The models with cracks completely change the behavior of the dome. The internal hoop forces disappear. The dome behaves like an array of arches.

Mark and Hutchinson studied the cracked dome and modified their model by changing the boundary conditions. The model studied in this article has been created by dividing the base of the dome into twenty-four meridians. In this way, the internal hoop forces are eliminated.

4.3.1 Study carried out with Abaqus
Three models have been elaborated. In Model 7 the dome has been divided into twenty-four segments, with a length similar to that in Terenzio’s study.

This model is proposed with the sole purpose of showing the cracked dome. The lower part of the dome is divided into twenty-four slides (each fifteen degrees). Figure 6 shows the dome deformed under its own weight.

In Model 8 the dome has a pier at the uppermost level, with light concrete. In Model 9 the stepped-rings are added to Model 8.
Table 3. Internal hoop and meridional stress at the base of the dome and on the interior of the copula (kg/cm²).

<table>
<thead>
<tr>
<th></th>
<th>Model 7 Light density</th>
<th>Model 8 Light density</th>
<th>Model 9 Light density</th>
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<tbody>
<tr>
<td></td>
<td>HF</td>
<td>MF</td>
<td>HF</td>
</tr>
<tr>
<td>Mark &amp; H</td>
<td>*</td>
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</tr>
<tr>
<td>FEA (base)</td>
<td>*</td>
<td>−16.7</td>
<td>*</td>
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<tr>
<td>FEA (1)</td>
<td>*</td>
<td>−4.85</td>
<td>*</td>
</tr>
<tr>
<td>FEA (2)</td>
<td>*</td>
<td>−3</td>
<td>*</td>
</tr>
<tr>
<td>FEA (3)</td>
<td>*</td>
<td>−1.15</td>
<td>*</td>
</tr>
</tbody>
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*Hoop stress does not exist in this part of the dome.
**M & H do not give values.

HF = Hoop Forces, MF = Meridional Forces.

The results are shown in table 3. The values are taken in three points (interior surface of the dome) coinciding with the rings and at the base of the dome.

4.3.2 Study carried out with graphical analysis

The stability or equilibrium approach is the most important concept to assess the safety of these structures. Equilibrium can be visualized using a line of thrust, the theoretical line or inverted catenary, which represents the path of the resultants of the compressive forces throughout the structure. The forces’ line passes through the pier up to the ground-line. At this point the Stability Factor (SF), or Rankine factor, is obtained (SF = b/2c, b being the pier weight). This factor was determined by W. J. Rankine in the mid-nineteenth century. If that factor takes a value 1, it indicates the collapse point of the pier: a value 3 is considered safe. That study was conducted with a Graphical Analysis method.

Various suppositions have been made. Firstly, two thrust lines were found for both models.

Figures 7 shows the two thrust lines found for each supposition of the dome profile of the Pantheon. The weight of the uppermost level is 13.10 N. Stability factors of SF8 = 2.9 and SF9 = 3.9 were obtained for model 8 and Model 9, respectively.

In order to know the influence of different densities used in the construction of the Pantheon, three variables were taken for Model 9 in function of the unit weight: a) the dome at 1,350 and 1,600 kg/m², b) the dome at 1,350 kg/m² (lighter) and c) the dome at 1,600 kg/m² (heavier). The stability factors obtained are: SF9 a = 3.9, SF9 b = 3.1 and SF9 c = 2.9. Those values are compared with the values obtained by Lancaster and are reflected in table 4.

4.3.3 Thrust line in a geometrical context

We can propose another question. What is the maximum height allowed for the Rotunda? What is the relation between geometry and the thrust line? Figure 8 shows different stability values for model 9 in function of the height of the wall. That factor is taken at the base of the pier and takes a value 1, indicating the collapse point of the pier. The actual height is a diameter of one sphere (73.75 Roman Feet) inscribed in its interior. If the high value increases 82.35 Roman Feet the SF is 1 and the thrust line passes out of the pier. If the high increases 27 Roman Feet the SF value is 2, where stability starts to be critical.

The thrust line is tangent to the extrados at point A. That point coincides approximately with the vertex of the so-called Diophantine triangle. Point A is
4.4 Model 10: full mode with arches

Earlier models show different suppositions about the construction of the dome. They study the influence of the different densities, the uppermost level, the stepped-rings and the geometry of The Rotunda. The next study has been done only with a numerical model. This model was created by modeling the arches built inside the wall and the lower part of the dome of The Rotunda. We do not have more details about their construction. Those arches distribute upper loads to the piers during the lengthy period in which the concrete is curing, but after curing, it becomes an integral part of the wall. This archway of bricks was only part of the wall and did not extend into the dome.

The question is: what role do the arches have in the construction of the rotunda? The big arches are built over the exedras, discharging them of the load. Figure 9 shows the main tensile stress. Those values are important to predict cracking. It is clear that the crown of each arch is supporting a greater stress. In this model we are assuming that is a rigid model and the dome are fixed over their supports. (Figure 9)

The maximum tensile stress is found over the arches, in the central voussoir, with a value of 4.2 kg/cm². It is interesting to see the main tensile stresses that act on the dome. Figure 10 shows that the dome and the others part of The Rotunda are hidden.

Figure 8. Relation between geometry and thrust line (model 9a).

Figure 9. Model 10, main tensile stress.

Figure 10. Model 10, main tensile stresses, only in the dome.
5 DISCUSSION

5.1 Results obtained with Abaqus

On the one hand, the results found here with numerical methods correspond to the same order of the values gathered by Mark and Hutchinson, with minor deviations. The values found are fairly consistent, with minor deviations that may be due to a large extent to differences in the mesh and type of elements used. Mark and Hutchinson do not point out where the maximum values given were measured. Those results validate our method.

On the other hand, those values have served as a base for comparing results obtained with graphical analyses. Then the results obtained in first Models 1–4 allow validation of the methods utilized and at the same time, give some information about the influence of different unit weights of the materials. The use of lighter-weight concrete towards the crown of the dome improves the structural behavior, reducing the internal hoop and meridional forces.

Models 5–6 show the influence of the stepped-rings and their different densities.

According to MacDonald, “the rings add to the load over the critical or launch portion of the great vault and function as buttresses, helping to bring the structure into stability through compression.” After that study, we can observe that the rings have been favorable for increasing the compression value, but the dome was working under compression before the step-rings came into play. Their function as buttresses is not clear. It is likely that the rings had a constructive function at the moment when the concrete is poured and spread.

The last model evaluated, Model 10, shows how the interior arches work with greater tensile stress in the central voussoir. The hoop forces reach a value of 9.4 kg/cm² and the meridional forces a value of −0.1 kg/cm². Those values indicate where and why the dome is cracked.

5.2 Results obtained with graphical analysis

The utilization of both methods allows a comparison of the information obtained. The graphical analysis utilized is a simple, intuitive, fast and easy method to implement. The three most important structural criteria for a masonry structure are: resistance, rigidity and stability. In the case of the Pantheon, the first two criteria are irrelevant because the deformations are very small. The third criterion, stability, is more relevant. The numerical method gives information about the first two criteria, but the graphical method gives information about the third criterion.

Those structures work under their own weight rather than under traction. Non-reinforced concrete works only under compression. They are hyper-static structures where numerous solutions exist. When the structure is analyzed with the graphical method, the objective will be to find a thrust line contained inside the thickness of the arch. At this moment we can say that this structure will work properly. The different thrust lines obtained show how the use of different densities benefits structural behavior. The whole dome at 1,350 kg/m³ gives a stability factor lower than if the dome were at 1,350 and the lower part at 1,600, improving it by 20%.

The stability factor obtained for different models varies from 2.9 to 3.9. If the dome had been built without the third level, the uppermost level, it would have been outside the limits of stability (Model 3, SF = 1.38), with the uppermost level Model 8 improving and doubling this value, resulting in a valid thrust line. This value increases to 3.9 with stepped-rings and can be considered safe and stable. The rings are apparently used for two purposes: for constructive reasons and for improving the structural behavior of the dome, but not as a buttress.

We assume that the dome is acting in a minimum thrust state, as reflected by the cracks. These cracks are irregular and narrower in shape and recorded as being “in situ” by Alberto Terenzio in 1930. The lower part or base of the dome is built similarly to The Rotunda, with an archway of bricks covered by concrete. That zone does not act as a cupola. The thrust line is tangent to the extrados at approximately 51.7° and in the intrados at 17°. The tangent points on the intrados coincide with the archway. The crowns of the arches are set at great tensile stress. The arches unload the stress on eight piers. It can be observed that the cracks coincide noticeably with the construction if the arches.

6 CONCLUSIONS

This paper presents a comparison between two methods of structural analysis. Those methods have been applied to ten different models of The Pantheon in Rome.

The results obtained have been compared to Mark’s results. The values gathered by Mark referred only to hoop forces. The deviations obtained were small. These studies confirm the numerical method utilized and offer additional information about the hoop and meridional forces, specifically the values of the meridional forces and the points of the dome where the values were taken. Models have been elaborated with greater precision than before.

The use of different densities in the dome improves the stability factor and reduces the values of both hoop and meridional forces. The stability factor increases by 20%. The studies performed with graphical analyses confirm the results obtained and give additional information about the stability of The Rotunda and the angle where the hoop forces change from compression...
to tension. It is confirmed that the uppermost level and stepped-rings improve the stability factor.

The cracks defined by Terenzio coincide with the position of the archway and with the points where the thrust line is tangent at the intrados and extrados of the cupola. There is a close relation between the geometry inherent to The Pantheon, the thrust line calculated for this geometry and the material used.

Lastly, it is concluded that the numeric model is more complete and informative, but still is a theoretical model and is based on numerous assumptions which can never be known. When the mesh is fine and the elements have many nodes, it requires powerful computers to post-process the information given. This method offers values for stress and deformations, but does not give information about stability. Nevertheless, the graphical model offers information about the stability of the dome and the position of cracks. Cracks can be simulated with numerical analysis, but require models of materials, information about material behavior, long procedures and non-linear analysis.

REFERENCES


