Vulnerability to natural hazards, preparedness and retrofitting

Structural Analysis of Historic Construction – D'Ayala & Fodde (eds)

Lower and upper bounds in closed form for out-of-plane strength of masonry structures with frictional resistances

C. Casapulla
Department of "Costruzioni e Metodi Matematici in Architettura", University of Naples Federico II, Naples, Italy

ABSTRACT: This paper presents a new simplified procedure to evaluate upper and lower thresholds for the possible collapse load factor for out-of-plane loaded block masonry structures with frictional resistances, by means of limit state analysis. Bypassing a detailed discrete element analysis, it is herein assumed that the general failure involves a number of cracks which separate the structures into a few *macro* blocks and all the possible relative motions among *micro* blocks are concentrated along the cracks. A particular class of mechanisms involving out-of-plane friction forces (i.e. interaction of torsion strength and shear forces on frictional interfaces) is analysed, and two limiting conditions for the collapse load factor are kinematically computed with simple formulations in closed form, by use of minimization routines. Sets of curves of the results in relation to the shape ratios of the unit and to the friction coefficient are then presented and widely discussed.

1 INTRODUCTION

In the light of the latest damaging seismic events which have occurred in many historic centres, increasing interest has recently been focused on the environmental vulnerability of the historic heritage buildings and monuments, with particular reference to the assessment of the seismic risk for buildings.

In this framework, many studies are based on the direct observation of recurrent damage and collapse mechanisms in seismic scenarios and are aimed at calculating the ultimate load factors by means of limit-state analysis. Therein, unreinforced masonry is generally modelled as an assemblage of discrete rigid blocks with frictional behaviour (Livesley 1978, Begg & Fishwick 1995, Baggio & Trovalusci 2000, Ferris & Tin-Loi 2001, Gilbert et al. 2003, Orduňa & Lourenço 2005, Gilbert et al. 2006).

Concerning the failure modes, besides the in-plane failure, the most recurrent failure mechanisms surveyed involve out-of-plane behaviour of masonry walls and such approaches in limit state analysis, using *macro* elements instead of discrete elements, are particularly interesting, especially when frictional behaviour is considered (Lagomarsino 1998, Casapulla 1999, Azevedo et al. 2000, De Felice & Giannini 2001, D'Ayala & Speranza 2003, Lagomarsino & Podestà 2004a, b, Casapulla & Maione 2007). However, the comparison between analytical and experimental results existing in literature reported by Restrepo-Vélez & Magenes (2005) highlights that the analytical ones generally overestimate the true collapse factors, both for in-plane and out-of plane failure mechanisms.

In this work, firstly, a likely reason for this discrepancy is suggested, according to what has already published in recent papers (Casapulla & D'Ayala 2006, Casapulla & Maione 2007). Then, using a *macro* element model and limit state analysis, a new computational strategy is presented aimed at identifying upper and lower thresholds for the possible collapse load factor by considering how friction forces develop along the crack lines. The procedure is herein tested with reference to the particular class of out-of-plane mechanisms involving the torsion-shear interactions shown in Figure 1 and then later explained.

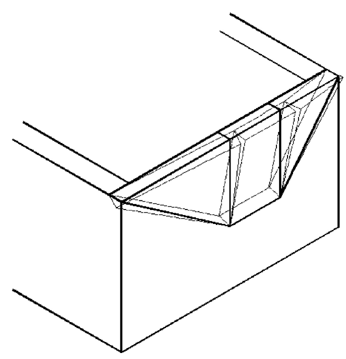

Figure 1. Failure mechanism characterised by out-of-plane frictional resistances.

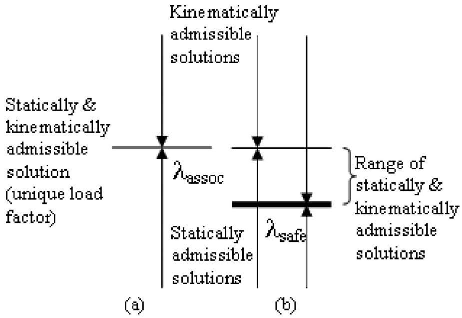

Figure 2. Static and kinematic admissibility of (a) associative, and (b) non-associative solutions.

2 THE NEW SIMPLIFIED METHOD FOR BOUNDING THE LOAD FACTOR

2.1 *The starting point*

In order to account for the discrepancy between analytical and experimental results mentioned above, it has to be recognised that the adopted model of dry rigid block masonry with frictional interfaces, whether discrete or *macro*, has non-standard behaviour due to non-associative frictional sliding. Additionally, for this kind of material, the bounding theorems of plastic limit analysis do not in general provide unique solutions for the collapse load factor. In other words, there exists a range of solutions that are both statically and kinematically admissible (Fig. 2).

However, if a kinematic approach is adopted, the minimum load factor so obtained will also be the lower one of this range and, therefore, a safe solution. Thus the solution to the non-associative problem, satisfying the kinematic conditions, represents the likely “exact” value of the collapse load factor. Therefore, the overestimations of the analytical results, with respect to experimental ones, should be found elsewhere.

2.2 *The rationale behind the new strategy*

The actual reasons for the discrepancy between analytical and experimental results introduced above have already been presented in recent works for in-plane (Casapulla & D’Ayala 2006) and both for in-plane and out-of-plane failure mechanisms (Casapulla & Maione 2007) and are based on the concept explained below.

The numerical procedures existing in literature generally account for the activation of frictional resistances on all contact surfaces crossed by the generic cracks, which separate the structure into a number of *macro* blocks. Actually, the activation of them, or part of them, on contact surfaces will be strictly related to the type of mechanism that occurs. In fact, for pure sliding to take place, the friction forces of all courses will be activated, and for pure rocking to initiate, a clean separation surface with no friction will occur, while for mechanisms with a general combination of sliding and rotation among blocks, it is not easy to identify the number of active sliding interfaces along the crack. This means that, for a general and more realistic combination of failure modes, the actual frictional resistance activating along the crack is smaller than that generally computed, and is quite unlikely to be estimated accurately.

A recent work of the author (Gilbert et al. 2006) presents a robust computational strategy for the “exact” ultimate load factor for in-plane loaded *micro* block masonry walls in the presence of non-associative friction, within a discrete element analysis approach. The iterative procedure involves solving a series of linear programming sub-problems rather than working directly with computationally expensive non-linear programming problems as others have done (Begg & Fishwick 1995, Baggio & Trovalusci 2000, Ferris & Tin-Loi 2001, Orduña & Lourenço 2003). Elsewhere (Casapulla & D’Ayala 2006), it was proved that the so computed “exact” values for in-plane problems are actually smaller than those calculated by using *macro* element analysis accounting for full development of friction forces along the cracks.

This means that the reason for the discrepancy above lies in the transfer from the *micro* to *macro* scale of the model of rigid blocks, where information about the actual active sliding interfaces is lost.

2.3 *The new simplified procedure*

In general, for a model considering a constitutive law of pure contact within the *micro* rigid blocks governed by friction, there are several possible relative mechanisms between each couple of blocks. One is pure sliding with relative displacements (in-plane and out-of-plane) tangential to the interfaces. Another is pure separation with relative displacements normal to interfaces. A third is pure rotation (in-plane and out-of-plane and Heyman’s type) and the final is pure twisting about the centroid of the interfaces. There may also be combination of these modes.

In order to simplify the analysis, it is herein assumed that the general failure involves a number of cracks, which separate the structures into a few *macro* blocks, and all the possible relative motions among *micro* blocks are concentrated along those cracks. Bearing in mind that the transfer from *micro* to *macro* scale of the model leads to a loss of information as described above, a maximum and minimum (zero) frictional resistance can be associated with the generic crack line regardless of the specific failure mode occurring. This means that a yield criterion for each crack line could be proposed, without reference to the type of mechanism activated.

On the basis of the assumptions above, a range of existence of the collapse load factor that bounds such solutions obtained, either with the discrete element analysis approach or with experimental testing, can now easily be defined in closed form. The procedure suggested implies the following steps:

1. the upper bound of the collapse load factor and the corresponding crack pattern are computed, under the assumption that maximum friction will occur on all contact surfaces crossed by the cracks;
2. removing the friction, once the crack pattern has been identified, a minimum value of the collapse load factor, depending only on geometric parameters, can also be defined.

In other words, the present simplified method is based on the fact that, whatever the type of relative mechanisms between each couple of blocks on opposite face of the critical cracks, the collapse load factor is bound from above by the assumption of full development of the friction force on every contact surface and from below by the total absence of friction. Therefore, with reference to the chosen out-of-plane mechanism in Figure 1, firstly the frictional resistances along the crack lines are evaluated, and then the two described limiting conditions can be found by using standard minimisation routines. This simplified procedure has already been developed in recent works for in-plane (Casapulla & D'Ayala 2006) and for both in-plane and other out-of-plane failure mechanisms (Casapulla & Maione 2007).

3 ASSESSMENT OF THE OUT-OF-PLANE FRICTION FORCES

The frictional resistances arising along the vertical and diagonal crack lines characterising the mechanism in Figure 1 are represented by interactions of shear forces and torsion moments. The results obtained by the author in a previous paper (Casapulla 1999) will herein be revised. Those results were also reached later by others (Orduňa & Lourenço 2005).

In that work (Casapulla 1999) a preliminary study of the possible mechanisms on the contact interfaces between two *micro* blocks led to the assessment of the frictional forces of pure shear strength and pure torsion strength and their interaction. The mechanism associated with this interaction consists of a relative rotation about the twisting centre and the solution of the problem was found by defining equilibrium equations, taking into account the coordinates of the instantaneous centre of rotation.

Now, let l, b and h be the width, the thickness and the height of a single block respectively and let $s = l/2$ be the overlap between two units, as shown in Figure 3.

According to Casapulla (1999), the frictional resistances of pure shear force and pure torsion moment

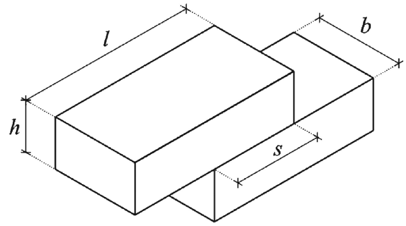

Figure 3. Block dimensions.

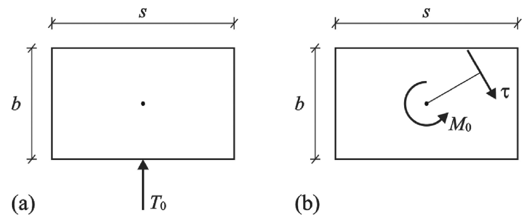

Figure 4. Frictional contact surface between two blocks. Pure shear force (a) and pure torsion moment (b).

on the generic contact surface can be written in the following respective forms (Figs. 4a, b):

$$T_0 = \tau bs = \gamma bshf \tag{1}$$

$$M_0 = T_0 d_0 \tag{2}$$

where τ is the uniformly distributed shear stress, γ is the specific weight of the material, f is the friction coefficient and:

$$d_0 = \frac{1}{12sb}\left[\begin{array}{l} s^3 \ln\dfrac{b+\sqrt{s^2+b^2}}{s} + 2sb\sqrt{s^2+b^2} + \\ + b^3 \ln\dfrac{s+\sqrt{s^2+b^2}}{b} \end{array}\right] \tag{3}$$

while the torsion-shear interaction adopted in favour of safety is:

$$M = M_0\left(1-\frac{T}{T_0}\right) \tag{4}$$

where T is the shear force acting on the block.

4 COLLAPSE LOAD FACTOR AND FAILURE MECHANISM

The goal of this section is to evaluate the range of existence of the collapse load factor for the mechanism in Figure 1. This represents the failure of the façade wall constrained at two sidewalls and loaded by the self-weight and horizontal out-of-plane forces proportional

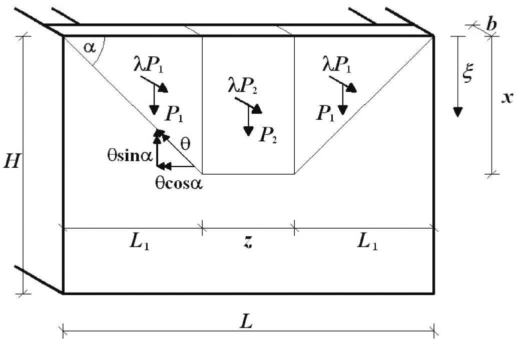

Figure 5. Geometry for the collapse mechanism.

Figure 6. Tested specimen (Restrepo-Vélez & Magenes 2005).

to the weight, simulating seismic actions. The latter can be oriented both inward and outward, with the assumption that, if outward, the overturning of the entire wall, due to the yielding of the sidewall capacity, is prevented. In order to be triggered, the mechanism must also satisfy the condition whereby the resistance to the outward displacement, developed by the edges of the façade, is overcome. That is to say, the arch effect in the façade wall is negligible.

The crack pattern of this class of mechanisms is characterised by a central trapezoidal portion of the wall, consisting of two triangular and one rectangular *macro* blocks rotating about cylindrical hinges (horizontal, vertical and diagonal), which identify the crack lines. The friction forces arising along the vertical and diagonal cracks are represented by interactions of shear forces and torsion moments, as mentioned above. By assessing these resistances in function of the horizontal loads proportional to the weights by the factor λ, its upper bound can be found by means of the standard virtual work and differentiated, with respect of the height of the trapezoidal portion, in order to find the value of the latter, which yields the minimum value of λ. Once the crack pattern has been identified, a minimum value of the collapse load factor, depending only on geometric parameters, can also be defined by removing the friction.

To proceed with the analysis, Figure 5 is proposed here as a more accurate representation of the geometry of the collapse mechanism, in which the three *macro* blocks of the crack pattern separated by the cylindrical hinges have variable height x.

The inclination of the two diagonal cracks generally follows the line defined by the shape ratio of the units ($h/s = \tan\alpha$), as clearly derived by Figure 6, which sketches a picture of this kind of mechanism tested by Restrepo-Vélez & Magenes (2005). Thus, the width z of the central *macro* block is linked to the variable x by the relation:

$$z = L - \frac{2xs}{h} \tag{5}$$

Rotation θ of the diagonal hinges can be sketched as its two components: 1) $\theta \sin\alpha$ is the rotation about a vertical axis, which activates the work of torsion moments; 2) $\theta \cos\alpha$ is the rotation about a horizontal axis, which does not provide internal work (Heyman's type). As for the compatibility conditions, it is easy to derive that the rotation of the horizontal hinge is $\theta \cos\alpha$ and the relative rotation of the vertical hinges is $\theta \sin\alpha$. Besides, the *macro* blocks are subjected to their own weights (P_1 e P_2) and to the proportional horizontal actions (λP_1 e λP_2), all defining the external loadings.

At the current distance ξ from the top of the wall, Equations 1 and 2 in the previous section will become respectively:

$$T_{0\xi} = \tau_{\xi} bs = \gamma \xi bsf \tag{6}$$

$$M_{0\xi} = T_{0\xi} d_0 \tag{7}$$

where d_0 is given by Equation 3.

Hence, having defined the following parameters of weight:

$$P_1 = \frac{\gamma L_1 bx}{2}; \qquad P_2 = \gamma bzx \tag{8}$$

Equation 6 can be rewritten in the form:

$$T_{0\xi} = \frac{2P_1 sf}{L_1 x}\xi \tag{9}$$

where $L_1 = (L - z)/2$.

As for the torsion-shear interaction along the diagonal cracks, the mechanism associated with this consists of a relative rotation about the instantaneous centre C_1 (other than interface centre C and variable with the

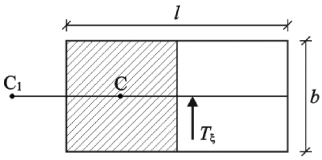

Figure 7. Frictional contact surface between two blocks along the crack. Shear-torsion interaction.

courses), as shown in Figure 7. The torsion moment at ξ is simply obtained from Equation 4:

$$M_{i\xi} = M_{0\xi}\left(1 - \frac{T_{i\xi}}{T_{0\xi}}\right) \tag{10}$$

where $i (i = 1, 2)$ identifies either diagonal hinge 1 or vertical hinge 2 and $T_{i\xi}$ is the share of the shear force on hinge i at ξ, in equilibrium with the external horizontal loadings.

The shear forces and torsion moments are assumed to vary linearly with x, since it is well known that the value of the friction force on a given surface is proportional to the weight of wall above it. Hence, as h is relatively small compared to H, the shear forces on the hinges per unit of height can be expressed in function of the external actions as:

$$t_{1\xi} = \frac{\lambda(2P_1 + P_2)}{x^2}\xi; \qquad t_{2\xi} = \frac{\lambda P_2}{x^2}\xi \tag{11}$$

so that $T_{i\xi} = h\ t_{i\xi}$ and the two moments for unit of height are obtained from Equation 10 as:

$$m_{1\xi} = \frac{M_{1\xi}}{h} = \left(\frac{2P_1 sf}{L_1 h} - \frac{\lambda(2P_1 + P_2)}{x}\right)\frac{d_0}{x}\xi \tag{12}$$

$$m_{2\xi} = \frac{M_{2\xi}}{h} = \left(\frac{2P_1 sf}{L_1 h} - \frac{\lambda P_2}{x}\right)\frac{d_0}{x}\xi \tag{13}$$

The internal virtual work can now be written as:

$$\begin{aligned} L_{\text{int}} &= \int_0^x 2(m_{1\xi} + m_{2\xi})\theta \sin\alpha\, d\xi = \\ &= \left[\frac{2P_1 sfx}{L_1 h} - \lambda(P_1 + P_2)\right]2d_0\theta\sin\alpha \end{aligned} \tag{14}$$

while the external virtual work is:

$$L_{\text{ext}} = \left(\frac{2\lambda P_1}{3} + \frac{\lambda P_2}{2} - \frac{P_1 b}{x} - \frac{P_2 b}{2x}\right)L_1\theta\sin\alpha \tag{15}$$

Thus, the virtual work equation for incipient collapse can be solved for the load factor λ in closed form:

$$\lambda = \frac{3\left[bL_1^2 h(2P_1 + P_2) + 8P_1 sfd_0 x^2\right]}{\left[(4P_1 + 3P_2)L_1 + 12d_0(P_1 + P_2)\right]xhL_1} \tag{16}$$

which, accounting for Equation 5 and the following adimensionalised parameters:

$$B = \frac{b}{l};\quad C = \frac{s}{h};\quad D = \frac{L}{l};\quad I = \frac{d_0}{b};\quad Y = \frac{x}{h} \tag{17}$$

can easily be rewritten in the form:

$$\lambda = \frac{3B[4IYf + C(2D - Y)]}{6BI(4D - 3Y) + 3DY - 2Y^2} \tag{18}$$

Within this Equation, B, C, D and f are the fixed parameters (I strictly depends on them) and Y is the variable parameter of the problem.

The minimum value of λ is yielded by:

$$Y = \frac{-2CD + \sqrt{A}}{4If - C} \tag{19}$$

where:

$$A = 4C^2D^2 - (4If - C)\left(6BCDI + 48BDI^2 f - 3CD^2\right) \tag{20}$$

Therefore, the upper threshold of the real collapse load factor is obtained from Equation 18 with its corresponding value of Y given by equation 19. Moreover, the lower threshold for the same crack inclination, i.e. the same value of Y, is simply obtained from Equation 18 setting the friction coefficient equal to zero.

If $z \to 0$, this mechanism tends to that analysed in a recent work by Casapulla & Maione (2007), where the shear forces $T_{2\xi}$ become negligible, and only pure torsion moments arise along the single central vertical crack.

5 DISCUSSION OF THE RESULTS

The following figures show the influence of the meaningful parameters of B, C, D and f, on the global resistance of the façade wall, with respect to the mechanism examined above. The wall height is irrelevant, provided that it is larger than x.

Above all, the width of the façade is the most important parameter. In fact, Figures 8 and 9 show that increasing the corresponding ratio $D = L/l$, the upper and lower thresholds for λ obtained by Equation 18 always decrease, and the normalised height of the trapezoidal portion involved in the mechanism proportionally increases, irrespective of the unit's shape ratio $B = b/l$.

However, while not strongly influencing the dimensions of the crack pattern, the unit's shape ratio clearly affects the global resistance. In fact, when D is taken fixed, the upper bound of λ (and consequently the

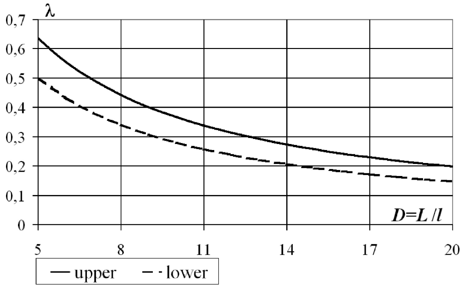

Figure 8. Upper and lower bounds for the collapse load factor λ versus wall's width ratio D, for $B = 0.5$, $C = 2$ and $f = 0.65$.

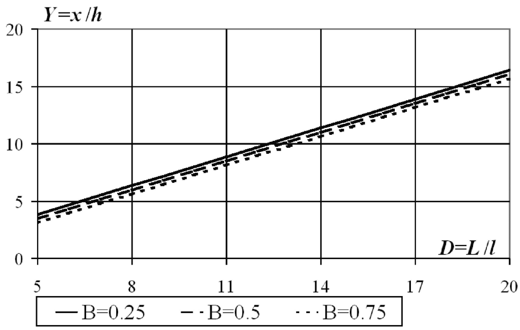

Figure 9. Height of the crack pattern normalized with h in function of the wall's width ratio D and of the unit's shape ratio B, for $C = 2$ and $f = 0.65$.

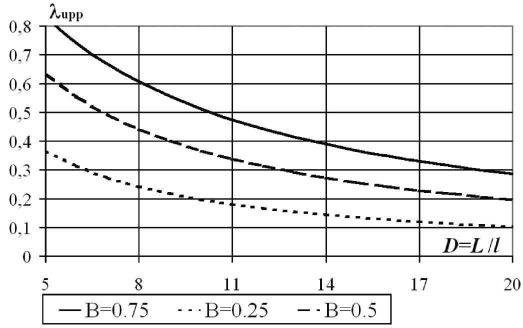

Figure 10. Curves of λ_{upp} in function of D and different values of B, for $C = 2$ and $f = 0.65$.

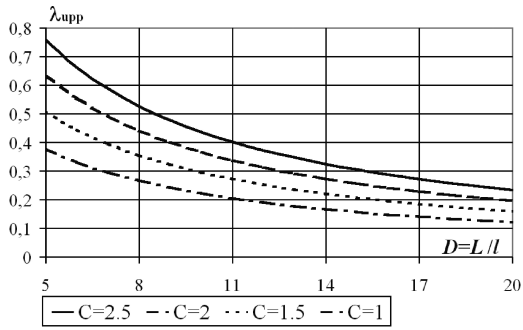

Figure 11. Curves of λ_{upp} in function of D and different values of C, for $B = 0.5$ and $f = 0.65$.

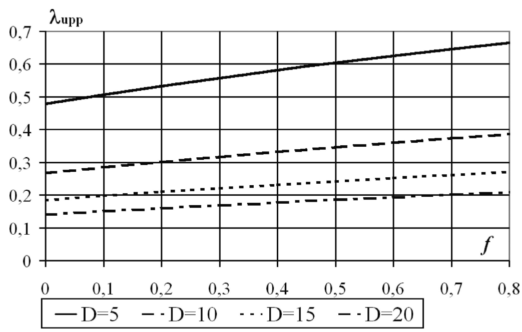

Figure 12. Curves of λ_{upp} in function of the friction coefficient f and D, for $B = 0.5$ and $C = 2$.

lower bound) decreases with decreasing values of the unit's shape ratio B, as shown in Figure 10. This again proves that thickset walls composed by thickset units in the horizontal plane present a larger contribution to the stability of such systems, as already stated for the mechanism with $z = 0$ (Casapulla 1999, Casapulla & Maione 2007).

Conversely, if the blocks are thickset in the vertical plane, the effect is detrimental. Indeed, keeping the same value of D, the upper bound λ_{upp} represented in Figure 11 decreases with the decreasing of $C = s/h$. This is due to the fact that reduction in C corresponds to reduction in the portion of the wall involved in the mechanism and therefore to reduction in its stabilising effect.

Lastly, Figure 12 shows that λ_{upp} is also influenced by the friction coefficient in an almost proportional way. However, this influence is quite irrelevant for slight changes of f, among the values generally assumed for the masonry structures. Instead, the width of the wall shows its great influence once again.

Finally, with reference to the mechanism examined in this paper and to different widths of the façade wall, in Table 1 the experimental results of specimens S1, S2, S5 and S6 published by Restrepo-Vélez & Magenes (2005) are compared with other existing analytical results already compared by the same authors and with the range of solutions λ_{upp} and λ_{low} herein obtained by means of the simplified method described. In particular, λ_1, λ_2 and λ_3 correspond to the results of D'Ayala & Speranza (2003), Picchi (2002) and Restrepo-Vélez & Magenes

Table 1. Comparison of analytical and experimental results, for $B = 0.5$, $C = 1.4$, $f = 0.7$ and variable D.

Test	D	λ_{test}	λ_{upper}	λ_{lower}	λ_1	λ_2	λ_3
S1	11	0.254	0.266	0.186	0.222	0.334	0.235
S2	11	0.226	0.266	0.186	0.222	0.334	0.235
S5	8	0.349	0.346	0.247	0.187	0.459	0.333
S6	13	0.208	0.230	0.160	0.279	0.283	0.183

(2005), respectively, while λ_{test} is referred to the specimens tested. It is worth noting that the results of the present author mentioned in the tables of Restrepo-Vélez & Magenes (2005) refer to the particular case of this mechanism when $z = 0$. Hence, these results, published in Casapulla (1999), cannot be compared directly with the others above. Moreover, it should be mentioned that the mechanism analysed by D'Ayala & Speranza (2003) is slightly different from this and, therein, the torsional effect of friction on the cylindrical hinges has been assumed negligible. This explains the underestimation of λ_1 in the table.

Specifying that test S2 is a second trial with the same specimen S1, from Table 1 it is clear that in all cases the computed bounds for the load factor strictly contain the experimental results, while λ_1 and λ_2 render errors up to around 50%, as already declared by Restrepo-Vélez & Magenes (2005). Besides, the new ad hoc formulation suggested by the latter for λ_3, which better fits the experimental results, retains a merely empiric aspect, but could be useful to get closer to the exact solution, as it moves from the bounds given by the present procedure.

6 CONCLUSIONS

The simplified method described in this paper permits the assessing of reliable ranges of existence of the collapse load factor for in-plane and out-of-plane loaded masonry walls. A particular class of mechanisms characterised by (out-of-plane) torsion-shear interactions on frictional interfaces has been analysed.

Bypassing a detailed discrete element analysis, it has herein been assumed that the general failure involves a number of cracks which separate the structures into a few *macro* blocks and that all the possible relative motions among *micro* blocks are concentrated along the cracks. Hence, upper and lower bounds for the collapse load factor have been computed, characterising the former by the assumption of full development of the friction force on every contact surface and the latter by the total absence of friction, respectively.

A standard kinematic approach within the limit state analysis has been used to obtain the two limiting conditions in closed form, only depending on the width of the wall, the unit's shape ratio and the friction coefficient. The results of the procedure outlined for the examined class of mechanisms are shown together with a discussion as to the most influential parameters and the correlations among them. Comparisons with experimental results have also been undertaken.

In order to assess the seismic risk for masonry buildings, the proposed procedure can easily be extended to other classes of mechanisms. Further developments will be presented in another work.

ACKNOWLEDGEMENTS

This research has been supported by Italian Civil Protection within the RELUIS Project (Line 1). The author is involved as a co-investigator.

REFERENCES

Azevedo, J., Sincrain, G. & Lemos, J.V. 2000. Seismic behaviour of block masonry structures. *Earthquake Spectra* 16(2): 337–365.

Baggio, C. & Trovalusci P. 2000. Collapse behaviour of three-dimensional brick-block systems using non-linear programming. *Struct. Eng. & Mech*. 10(2): 181–195.

Begg, D.W. & Fishwick, R.J. 1995. Numerical analysis of rigid block structures including sliding. *Comp. Meth. Struct. Masonry* 3: 177–183.

Casapulla, C. 1999. Resistenze attritive in una parete muraria soggetta ad azioni normali al suo piano medio. In *L'Ingegneria Sismica in Italia; Proc. IX nat. conf. Anidis, Torino, 20–23 September 1999 (CD-R)*. Torino: Fond. Crt. (in Italian)

Casapulla, C. & D'Ayala, D. 2006. In-plane collapse behaviour of masonry walls with frictional resistances and openings. In P.B. Lourenço et al. (eds.), *Structural Analysis of Historical Constructions (SAHC06); Proc. V intern. conf. New Delhi, 6–8 November 2006*: vol. 2, 1159–1166. New Delhi (India): Macmillan.

Casapulla, C. & Maione A. 2007. Procedura semplificata per meccanismi di danno di strutture murarie nel piano e fuori dal piano. In Plus (ed.), *L'Ingegneria Sismica in Italia; Proc. XII nat. conf. Anidis, Pisa, 10–14 June 2007 (CD-R)*. Pisa: Pisa University Press. (in Italian)

D'Ayala, D. & Speranza, E. 2003. Definition of collapse mechanisms and seismic vulnerability of masonry structures. *Earthquake Spectra* 19(3): 479–509.

De Felice, G. & Giannini, R. 2001. Out-of-plane seismic resistance of masonry walls. *Journal of Earthquake Engineering* 5(5): 253–271.

Ferris, M. & Tin-Loi, F. 2001. Limit analysis of frictional block assemblies as a mathematical program with complementarity constraints. *Int. J. Mech. Sci.* 43: 209–224.

Gilbert, M., Ahmed, H.M. & Casapulla, C. 2003. Computational limit analysis of masonry structures in the presence of non-associative friction. In T.G. Hughes & G.N. Pande (eds.), *Computer Methods In Structural Masonry (STRUMAS VI); Proc. VI intern. Symp., Rome, September 2003*: 1–8. Swansea (Wales): Computers & Geotechnics Ltd.

Gilbert, M., Casapulla, C. & Ahmed, H.M. 2006. Limit analysis of masonry block structures with non-associative frictional joints using linear programming. *Computers and Structures* 84(13–14): 873–887.

Griffith, M.C., Magenes, G., Melis, G. & Picchi, L. 2003. Evaluation of out-of-plane stability of unreinforced masonry walls subjected to seismic excitation. *Journal of Earthquake Engineering* 7(1): 141–169.

Lagomarsino, S. 1998. A new methodology for the post-earthquake investigation of ancient churches. In *Earthquake Engineering; Proc. XI European conf., Paris*: 67–78. Rotterdam: Balkema.

Lagomarsino, S. & Podestà, S. 2004a. Seismic vulnerability of ancient churches. Part 1: damage assessment and emergency planning. *Earthquake Spectra* 20(2): 377–394.

Lagomarsino, S. & Podestà, S. 2004b. Seismic vulnerability of ancient churches. Part 1: statistical analysis of surveyed data and methods for risk analysis. *Earthquake Spectra* 20(2): 395–412.

Livesley, R.K. 1978. Limit analysis of structures formed from rigid blocks. *Int. J. for Num. Meth. in Eng* 12: 1853–1871.

Orduňa, A. 2003. Seismic Assessment of ancient masonry structures by rigid blocks limit analysis. *Ph.D. Thesis, University of Minho*.

Orduňa, A. & Lourenço, P.B. 2005. Three-dimensional limit analysis of rigid blocks assemblages. Part I: torsion failure on frictional interfaces and limit formulation. *International Journal of Solids and Structures* 42: 5140–5160.

Picchi, L. 2002. Risposta sismica per azioni fuori dal piano di pareti murarie. *Graduation Thesis, University of Pavia*. (in Italian)

Restrepo-Vélez, L.F. & Magenes, G. 2005. A mechanics based methodology for the evaluation of the seismic risk of unreinforced masonry buildings, *Research Report no. ROSE-2005/05*. Pavia: IUSS Press.