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Influence of friction and tensile resistance on the stability of masonry arches

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ABSTRACT: The application of limit analysis to the safety assessment of block masonry structures usually presupposes absence of sliding between the blocks and joints without tensile resistance (Heyman 1966). The "standard" plasticity theorems can then be used to qualify situations either as "stable" or as "unstable".

Experience shows nonetheless that in some circumstances, in vaults in particular, sliding does occur. Moreover, the resistance to tension of the joints, even if it is often low, is usually not zero. The plasticity theorems consequently become inapplicable: masonry is not a "standard" material.

In case of limited friction and tension resistance, structural situations have then to be classified as "certainly stable", "maybe stable" and "certainly unstable". These three situations can be delimited by two stability domains G_M and G_m in the space of the actions (forces or displacements).

This paper presents techniques for the construction of G_M and G_m, first at the *material level*, then at the *joint level* and finally at the *structure level*. It shows how global information on the structure can be used to define more usable local stability domains. For clarity, techniques are explained using an elementary structure but the methodology presented is sufficiently general to be applicable to more complex block masonry structures.

1 INTRODUCTION

The most perturbing consequence of the "non-standard" character of masonry is that the apparently reasonable assumption (a corollary of the plasticity theorems) that "if it can be proved that a model which is everywhere weaker than the real structure under study is stable under a given set of forces, then the real structure will necessarily be stable" becomes invalid.

This is perturbing because the above corollary has two invaluable benefits:

1. To study a "real structure", it is not necessary to build a complex model trying to reproduce it very accurately. A simpler model can be used as long as it is in every point weaker.
2. It is not necessary to search for the "real state of stress" in the structure. If it is possible to find any statically admissible state of stress in the model, then the "real structure" is safe. (Heyman 1966).

This possibility is invaluable because ancient masonry structures are typically hard to qualify, resulting in many uncertainties (on shape, deformations, materials, loading history, construction technique, homogeneity, etc) and the corollary of the plasticity theorems (partly) saves the analyst from having to become Laplace's demon!

If the failure of this corollary has a direct influence on *limit analysis* of masonry structures, it has actually also an influence on the validity of analysis made with methods taking into account the deformation of the materials (e.g. finite element analysis).

Surprisingly, the suspicion that the "non-standard" character of masonry (already recognised by Drucker (Drucker 1954)) is casting on the validity of structural analysis of masonry is not very often discussed. There are exceptions (Boothby and Brown 1993; D'Ayala and Casapulla 2001a, D'Ayala and Casapulla 2001b; Smars 2000, and others) but too many engineers are probably still unaware or ignore the problem. Others may only rely on their engineering judgement. The practical consequence is that, in some particular cases, standard assumptions actually lead to overestimates of the safety level.

One possible approach to the problem is through statistical methods (Schueremans et al. 2001), analysing the stability of the structure taking into account the actual distributions of material properties, geometrical parameters, external forces and, very importantly, displacements of the supports. This is certainly complex and it appears that a limit analysis approach is still beneficial.

Three elementary examples are introduced to illustrate the failure of the "standard" plasticity

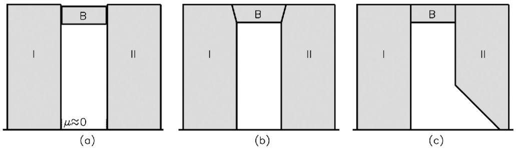

Figure 1. Three elementary structures.

theorems and to clarify the intention of the paper (Figure 1).

The structure of Figure 1a is made of two piers (*I* and *II*) holding a block *B* by friction. The piers are massive and the block is rather small; it is therefore straightforward to find a statically admissible state of stress in the structure. Nevertheless, the structure is not safe; the normal force *N* in the joints between piers and block is not known and could vanish following any small perturbation (displacement of a pier, temperature variation), resulting in an important effect: the fall of block *B*.

The structure of Figure 1b is more realistic. It respects *standards of good practice* for the construction of lintels (masons are well aware of the "non-standard" character of masonry). Block *B* has a prismatic shape which prevents it from sliding. The structure is safe.

The structure of Figure 1c is identical to the structure of Figure 1a at the exception of the corner of pier *II* which was removed. It is everywhere locally weaker than the original structure and, if the above-mentioned corollary was valid for structures where sliding can occur, structure *c*. would be less safe than structure *a*. But this is not the case. The instability of pier *II* by itself guaranty the existence of a minimal force N_{min} on the joint sufficient to maintain block *B* in equilibrium. The structure is safe.

Note that, if the maximum tension resistance of the joints was high, the structure may then become unsafe: the joint at the base of pier *I* could be intact and the joints between piers and blocks could be cracked. Again, any small perturbation would then cause collapse.

The examples above may appear theoretical but they show clearly the type of problem and the directions to follow to define a safe domain.

In practice, various factors can increase the risk of sliding in masonry structures.

- Level of hyperstaticity of the structure (vaults are more in danger than arches, multiring arches than single ring arches (Melbourne1998 and Gilbert 1998)) (incidentally, it is not by accident that most theoretical developments concerning "non-standard" behaviour of materials have taken place in field of soil mechanics).
- Possible displacements of the abutments (breaking the structure into pieces forming a mechanism).

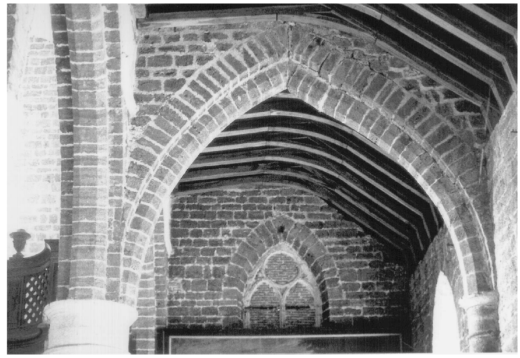

Figure 2. Saint-Catherine, Diest (Belgium), arch of which some voussoirs have slided.

- Double curvature of vaults (in case of displacement of the abutments, cracks will open more widely and the displacement of ribs and webs may be incompatible).
- Size of the structure (smaller structures are more influenced by the tension resistance of the joints).
- Fragility and relative high resistance to tension of the joints.
- Degradation of the mortar.

Figure 2 shows an example of an existing masonry arch of which some voussoirs have slided. This is not an exceptional case. During the preparation of his PhD dissertation (Smars 2000), the author visited many Gothic churches in the Belgian province of Brabant and encountered a significant number of masonry structures presenting sliding (particularly in vaults with a double-curvature and in diaphragm arches).

2 PROBLEM DEFINITION

Any model used to assess the safety of a structure is built on a set of hypotheses. The beauty of "standard" plasticity analysis is that powerful conclusions can be deduced from a few simple assumptions. Those assumptions may nevertheless be unsafe in some circumstances. In this paper, the influence of the release of two hypotheses: "no sliding of the blocks" and "no resistance to tension" is investigated.

The following hypotheses will be made:

1. The structure is made of blocks in contact through joints.
2. Blocks are rigid and infinitely resistant.
3. Joints have a finite resistance characterised by two domains of resistance (defined in section 4).
4. The shape of the structure is fixed (deformations resulting from changes in the external loads, movements of the supports, dilatation following changes of temperature are not. considered).

As the hypotheses of "standard analysis" (Heyman 1966), none of the previous hypotheses is safe. They are nevertheless less restricting and can therefore extend the domain of validity of "limit analysis" and bring insight on the limits of classical theory.

There is no doubt that more generality could be introduced studying the release of other hypotheses. Elsewhere (Smars 2000), the influence of finite displacement of the abutments of an arch (hypothese 4) was investigated.

On the base of those hypotheses, two domains will be defined: a "potentially stable" domain G_M and a "certainly stable" or safe domain G_m. The problem will be first tackled at the *material level* (section 3), then at the *joint level* (section 4) and finally at the *structure level* (section 5). The most original elements of the paper are to be found in section 5.

3 MATERIAL LEVEL

Materials composing the joints are characterised by a maximum compressive resistance σ_c and a maximum tensile resistance σ_t. Those values can be bounded and two domains defined: $G_M^m(\sigma) = [\sigma_{Mc}, \sigma_{Mt}]$ and $G_m^m(\sigma) = [\sigma_{mc}, \sigma_{mt}]$ (the superscript m refers to "material", the subscript M to "maximum" and the subscript m to ''minimum"). It is assumed that the material certainly does not resit stresses $\sigma \notin G_M^m$ and certainly resists stresses $\sigma \in G_m^m$. The low ductility of the material in tension imposes that $\sigma_{mt} = 0$.

With the exception of σ_{mt}, it may be difficult to assign values in specific cases. There is nevertheless no doubt that bounds exist and that investigations on the structure (detailed surveying, testing) can improve the confidence that the assigned bounds approach the real values.

Note that, to assign a non null value to σ_{mt}, it would be necessary to assume that no joint is cracked and that movements of the supports remain very small. Both assumptions are highly open to criticisms and difficult to control.

In most circumstances, the compressive resistance (σ_{Mc} and σ_{mc}) is probably not critical (Heyman 1966), especially if σ_{Mt} is low.

The most (only) critical value to assign is σ_{Mt}. As it will be shown, it has a direct influence on the definition of the "potentially stable" domain G_M^S of the structure S and an indirect influence on its "safe domain" G_m^S.

4 JOINT LEVEL

As for the *material level*, two domains will be defined at the *joint level*. For each joint i: a "potentially stable" domain G_M^i and a "safe domain" G_m^i are defined in the space of the stress resultants X^i on the joint. In 2D, X^i can for instance be decomposed in a normal force N^i, a tangential force Q^i and a moment M^i (usual beam convention). $X^i = (N^i, Q^i, M^i)$. Different stress distributions can produce identical stress resultants and the most favourable distribution should be used for G_M^i and the most unfavourable distribution for G_m^i.

4.1 *Definition of G_M^i*

The projection of G_M^i in the space $N \times Q$ is defined by the law of coulomb with an angle of friction φ_M^i, maximal normal forces $N_{Mt}^i = A^i\sigma_{Mt}$ in tension and $N_{Mc}^i = A^i\sigma_{Mc}$ in compression (where A^i is the area of joint i).

$$|Q^i| - (N_{Mt}^i - N^i)\tan\varphi_M^i \leq 0 \tag{1}$$

$$N^i \geq N_{Mc}^i \tag{2}$$

The projection of G_M^i in the space $N \times M$ has a more complex shape.

If any stress distribution is possible on the section, the domain is straightforward to define. The limit stress state is a uniform compression stress σ_{Mc} on one side of the section and a uniform tensile stress σ_{Mt} on the remaining of the section. The resulting domain is a double-parabola. This kind of distribution is commonly used for ductile materials like steel but is extremely unlikely for masonry due to its lack of ductility in tension.

If the (mild) hypothesis is made that stress distribution must result from linear distribution of deformations, possibly following a complex load history, then the domain is smaller. It can be shown (Smars 2000) that G_M^i is the union of three (possibly four) elementary domains (Figure 3).

$$G_M^i = G_{Me}^i \cup G_{Mc}^i \cup G_{Mt}^i (\cup G_{Mh}^i) \tag{3}$$

G_{Me}^i is a diamond-shape domain corresponding to an elastic distribution of stresses in the section.

G_{Mc}^i is a parabolic domain corresponding to a uniform state of compression stresses (see for instance Heyman 1998, p. 94).

G_{Mt}^i is a parabolic domain corresponding to a uniform state of tension stresses. This domain is the equivalent of G_{Mc}^i in tension.

G_{Mh}^i is a parabolic domain only active if the tension resistance σ_{mt} is relatively large ($|\sigma_{Mc}| \leq 3|\sigma_{Mt}|$) (see Figure 3b). It corresponds to an elasto-plastic state of stress.

Note that G_M^i is not convex.

No consideration is made on the ductility of the material and some tensional states are very unlikely to occur in practice; in particular, states $X^i : X^i \in G_{Mt}^i, X^i \notin G_{Me}^i$.

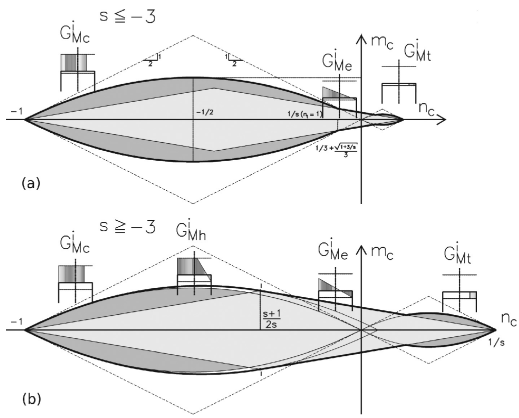

Figure 3. "Potentially stable" domain G^i_M of a joint with a rectangular section ($w \times d$) expressed in a normalised space ($n_c = \frac{N^i}{wd\sigma_{Mc}}$, $m_c = \frac{M^i}{wd^2\sigma_{Mc}}$). (a) small tensile resistance (b) high tensile resistance. (s is the ratio between compressive and tensile resistance $s \equiv \frac{\sigma_{Mc}}{\sigma_{Mt}}$).

4.2 *Definition of G^i_m*

The problems brought by friction are related to the fact that during sliding of blocks, the direction of movement is not perpendicular to the force $F^i = (N^i, Q^i)$ acting on the joint.

During the sliding of blocks, the angle between F^i and the normal to the joint is comprised between φ^i_m and φ^i_M.

If the surface of the joint is perfectly smooth, the direction of movement is parallel to the direction of the joint.

If the surface of the joint is not smooth (i.e. always), there will be some dilatancy: a displacement u parallel to the joint will induce a normal displacement v of the joint lips (usually an opening) related by the relation $v = u \tan \delta^i_m$ where δ^i_m is the angle of dilatancy.

Now, the normality of the flow rule is the key of the demonstration of the plasticity theorems. The absence of normality for frictional materials invalidate them when sliding can occur.

To define the projection of G^i_m in the space $N \times Q$, the first step is to define the domain G^{0i}_m of the statically admissible N and Q.

$$|Q^i| + N^i \tan \varphi^i_m \leq 0 \tag{4}$$

G^i_m can then be found as the envelope of the manifolds perpendicular to the directions of flow for all the limit states on the border of G^{0i}_m (Radenkovic 1961, Radenkovic 1962, Palmer 1966). By definition, it is a convex function. In the case of Coulomb materials with dilatancy, the resulting envelope is

$$|Q^i| + N^i \tan \delta^i_m \leq 0 \tag{5}$$

Note that the value of δ^i_m can be difficult to specify in practice and that, without dilatancy, the volume of G^i_m in the space $N \times M \times Q$ is void!

This is a disturbing observation as G^i_m will be used to define the safe domain of the structure. But it will be shown below that the "volume" of G^i_m can be increased taking into account contextual aspects.

Note also that any slight opening of the joint will prevent the positive effect of dilatancy. At the joint level it is therefore not safe to consider dilatancy. But again, contextual aspects will allow the reintroduction of an angle of dilatancy in some joints.

The projection of G^i_m in the space $N \times M$ is a parabolic domain corresponding to a uniform state of compression stresses σ_{mc}. In the space $N \times M$, it is reasonable to assume normality of the flow rule (Smars 2000, pp. 55–56).

5 STRUCTURE LEVEL

The n internal stress resultants in a hyperstatic structure S can be expressed in function of $m (m < n)$ parameters. They can be represented by a vector X_0. In the case of an arch, three parameters are necessary. One possibility is to choose the stress resultant on the first joint: $X_0 = (N^0, Q^0, M^0)$.

For arches, it is customary to express the safety domain by mean of a minimum thrust H_{min} and a maximum thrust H_{max} (Heyman 1966) (H is also function of X_0). Extending this idea to situations where tension resistance is possibly not null and where sliding can occur, a "potentially stable" interval $[H_{minM}, H_{maxM}]$ and a "certainly safe" interval $[H_{minm}, H_{maxm}]$ can be defined.

This is actually a special case of the problem of finding extrema of a particular function $F(X_0)$ defining the intervals $[F_{minM}, F_{maxM}]$ and $[F_{minm}, F_{maxm}]$. F can be for instance the force at a critical position.

The more general approach followed in this paper was inspired by a technique developed by Alfred Durand-Claye (Durand-claye 1867; Durand-Claye 1880) who defines domains of stability (*aires de stabilité*) for arches in the space of X_0. Two domains: G^S_M and G^S_m will be necessary in the framework of "non-standard" limit analysis.

From a practical point of view, the nonlinear programming problem of finding extrema of a function $F(X_0)$ can be easier to solve and implement.

5.1 *Definition of G^S_M*

$G^S_M(X_0)$ is the domain of *potential stability* of the structure S. Statical states $X^i(X_0)$ defined by X_0 : $X_0 \notin G^S_M(X_0)$ are impossible (this is the upper-bound theorem for frictional materials demonstrated by Drucker (Drucker 1954)).

G_M^S is a domain of "potential stability". The existence of a non-trivial G_M^S, i.e. of values $X_0 \in G_M^S$ does not guaranty stability of the structure.

G_M^S can be defined as the intersection of all the local joint domains G_M^i in the space of X_0.

$$G_M^S(X_0) = \cap_i G_M^i \left(X^i\left(X_0\right)\right) \tag{6}$$

5.2 *Definition of G_m^S*

It is of course much more useful to define a safe domain G_m^S, the existence of which guaranty the safety of the structure.

A first estimate of G_m^S could be found using the same method as for G_M^S.

$$G_m^S(X_0) = \cap_i G_m^i(X^i(X_0)) \tag{7}$$

Unfortunately, the resulting domain is void (as it is impossible to consider the influence of dilatancy at this stage: any joint can have an open crack).

Because the joint is part of a larger structure, new contextualised local domains G_m^{*i} with $G_m^{*i} \subseteq G_m^{0i}$ can be defined from which the safe domain of the structure can be derived using the same technique of intersection of the local domains.

$$G_m^S = \cap_i G_m^{*i} \left(X^i\left(X_0\right)\right) \tag{8}$$

The size of the local safety domain can be increased by two means: considering the existence of a "minimal normal force on the joints" and considering the influence of "stereotomy and dilatancy".

Minimum normal force

If a lower bound N_m^i can be found for the normal force N^i acting on a joint i, then

- A lower bound can also be fixed on the resistance to tangential force Q and the safe domain can be extended (an idea already expressed in the framework of soil mechanics (de Josselin de Jong 1964)) and used to demonstrate the stability of masonry domes (D'Ayala and Casapulla 2001a).
- The joint is necessarily closed and dilatancy of the joint surface can be reintroduced.

The determination of N_m^i for each joint is done in several steps.

Of course, N_m^i cannot be chosen as the minimum normal force on the joint from the set of all the forces for which the structure is stable: $\min_{X_0} N^i(X_0) : X_0 \in G_M^S$. If it was the case, the structure of Figure 4a would be stable.

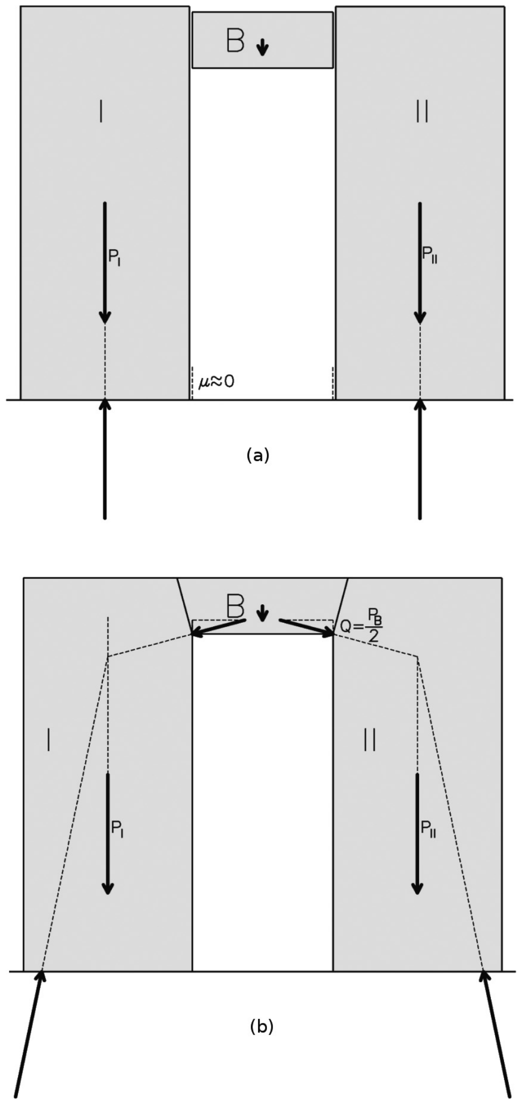

Figure 4. (a) Structure made of two piers (I and II) "holding" a block (B) by friction. Situation just after a small displacement u of the piers, annulling the normal force in the joints and consequently provoking the fall of B (b) Influence of stereotomy. Statical state completely inside the safe domain G_m^i (the force is perpendicular to the joint).

Figures 4 and 5 can actually be used to clarify the situation. If block B is removed from the structure, what force is necessary to maintain the remaining structure stable? In the case of Figure 4a, it is clearly 0; the two piers do not require forces on the joints to be stable. It is therefore possible to have situations in which the normal force is null and other situations in which it is just sufficient to maintain block B in equilibrium at the mercy of any small perturbation.

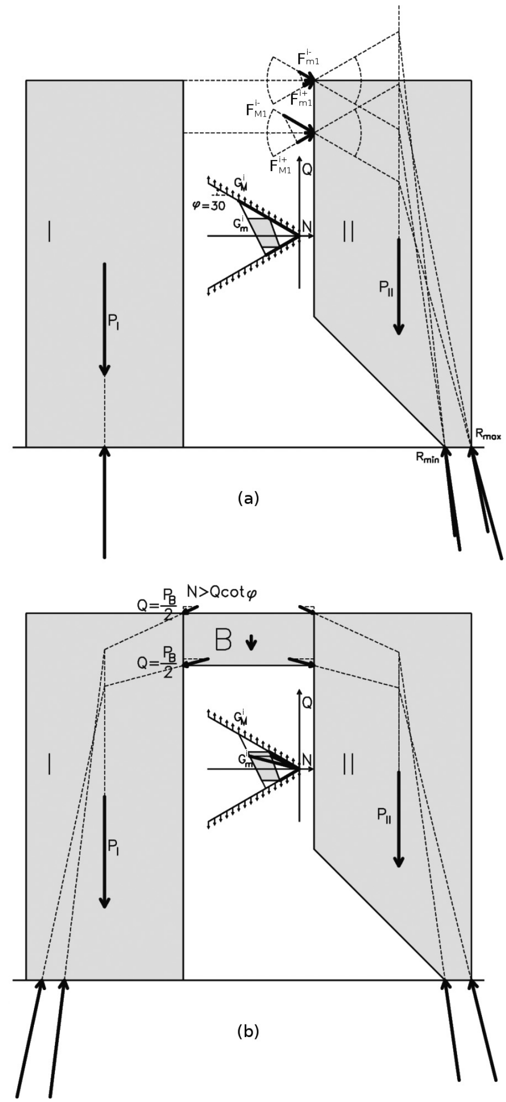

Figure 5. Influence of the normal force N active on the joints. The safe domain G^i_m is shaded. The arrows on the border of G^i_M indicate the flow direction. (a) Minimal and maximal forces compatible with the stability of pier II. (b) Statical state completely inside the safe domain G^i_m.

In the case of Figure 5a, pier II requires a force on joint i between B and II to be stable. N^i_m can therefore be determined.

N^i_m must necessarily be compatible with the stability of $S-B$.

$$N^i_m(X_0) \in G^{S-B}_M \tag{9}$$

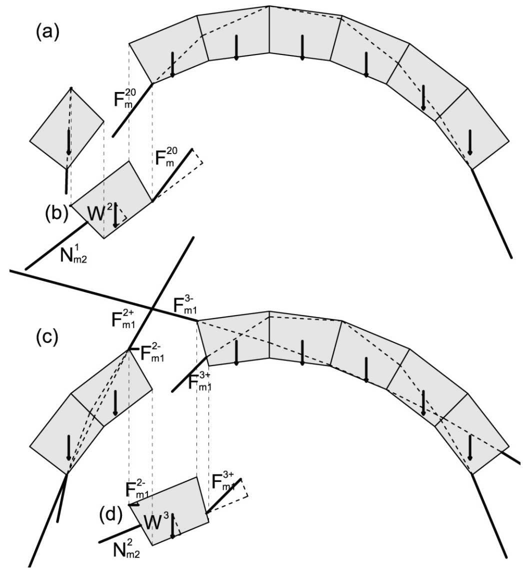

Figure 6. Simple arch $(\sigma_{Mt}=\sigma_{mt}=0, \varphi_m=\varphi_M=30°)$. (a) Analysis of the structure when the second block is removed (φ always $<30°$). (b) Determination of N^1_{m2}. (c) Analysis of the structure when the third block is removed. (d) Determination of N^2_{m2}.

In general, N^i_m does not need to be the minimum N^{i0}_m of the normal forces compatible with the "potentially safe" domain of the remaining structure.

$$N^{i0}_m = min_{X_0} N^i\left(X^i(X_0)\right) : X^i \in G^{S-B}_M \tag{10}$$

If the joint is unstable, it needs to slide and therefore, the angle of application of the force on the joint is imposed, either in the interval $[\varphi_m, \varphi_M]$, either in the interval $[-\varphi_M, -\varphi_m]$ and N^i_m can actually have two values N^{i+}_{m1} and N^{i-}_{m1} according to the sliding direction (the subscript 1 is used to indicate that it is a first estimate).

$$N^{i+}_{m1} = min_{X_0} \quad N^i\left(X^i(X_0)\right) : X^i \in G^{S-B}_M,$$
$$\varphi\left(X^i(X_0)\right) \in [\varphi_m, \varphi_M]$$

In some circumstances, it may be impossible to find one or any of those two values, i.e. the force necessary for the stability of $S-B$ possibly cannot have an angle of incidence belonging to one of the two intervals. In that case, $G^{*i}_m = G^{0i}_m$ (Figure 6a).

Using this technique of removal of a substructure, the minimal normal forces on a joint depends in fact of the substructure which is removed from the structure S. The substructure on one side of a joint which will lead to the lowest N^i on the joint is the adjacent block

which is therefore the substructure to consider for the definition of N_{m1}^{i+} and N_{m1}^{i+} on this side of the joint.

In the case of Figure 5b, this technique allows to find a non-zero N_{m1}^{i} on the second joint (between B and II) but not on the first (between I and B). Nevertheless, it is clear that if a normal force acts on one of the joint, it has also to act on the other joint.

To see how this or other forces can be used to define minimum forces on other joints, it is necessary to return to the block. In general, the resultant of the weight of the block and of all the minimal forces acting on the joints of the block is not null but even if the block is falling, during a short instant the component of the resultant in a direction tangent to the middle line of the arch or in a plane tangent to the middle plane of the vault has to be 0 (the displacement of B is partly constraint by $S-B$). It is therefore possible to find for each of the joint the minimum normal force N_{m2}^{i} to insure equilibrium in that direction/plane (Figures 6b and 6d).

The minimum normal forces on block B side of the joint are then defined as $N_m^{i+} = \max(N_{m1}^{i+}, N_{m2}^{i})$ and $N_m^{i-} = \max(N_{m1}^{i-}, N_{m2}^{i})$.

Those values have to be computed for all the blocks of the structure. At the end, each joint will be characterised by four values $N_m^{i+}(B^{k1})$, $N_m^{i+}(B^{k2})$, $N_m^{i-}(B^{k1})$ and $N_m^{i-}(B^{k2})$.

The steps necessary to find N_m^{i} for each joint can be summarised as follow

1. For each block B^k of the structure S find the domain of potential stability $G_M^{S-B^k}$.
2. For each joint j of block B^k find the minimal values of the normal force necessary for the stability of $S-B^k$: N_m^{j0} and (if they exist) N_{m1}^{j+}, N_{m1}^{j-}.
3. For each block B^k, find F^k, the component tangent to the middle line or the middle plane of the structure of the resultant of the weight of the block and of the N_m^{j0} acting on the joints of B^k.
4. For each joint of block B^k, find the minimal value N_{m2}^{j} necessary to equilibrate F^k. Define the the minimal value of $N_m^{j+} = \max(N_{m1}^{j+}, N_{m2}^{j})$ and $N_m^{j-} = \max(N_{m1}^{j-}, N_{m2}^{j})$
5. For each joint of the structure in contact with blocks B^{k1} and B^{k2}, define $N_m^{i+} = \min(N_m^{i+}(B^{k1}), N_m^{i+}(B^{k2}))$ and $N_m^{i-} = \min(N_m^{i-}(B^{k1}), N_m^{i-}(B^{k2}))$.

Using N_{min}^{i+} and N_{min}^{i-}, it is then possible to construct the safe domain G_m^{*i} (taking into account the dilatancy of the joints if $N_m^{i} \neq 0$).

Stereotomy and dilatancy

The positive influence of dilatancy was already discussed in section 4 but dilatancy is actually present at various levels.

The prismatic form of the voussoirs of an arch and the stereotomy of the stones in a vault can also trigger dilatancy in case of sliding. But contrarily to dilatancy in joints, the effect is only present in one of the sliding directions.

As for dilatancy in joints, a difference must be made between joints certainly in contacts (for which $N_m^{i} \neq 0$) and joints which are potentially open. To be able to consider an angle of dilatancy for those last joints, one has to be sure that the prismatic character of the joint is sufficient to accommodate a "finite" (to be defined) opening of the joints. If it is the case, this can have a positive influence on the joint dilatancy as well (bringing some "ductility").

The practical problem may be that in many structures, especially in vaults, the prismatic character of the block is too low especially if the joint thickness is taken into account.

6 CONCLUSIONS

A large class of problems can be solved efficiently with the "standard" methods of limit analysis but, in certain cases, sliding of blocks can occur and the confidence domain defined by such methods is not valid anymore.

In this paper, a technique is presented to build a "potentially stable" domain G_M and an estimate of a safe domain G_m for structures where sliding of blocks can occur and where the tensile resistance of joints is possibly not null.

Arguably, the main contribution of the paper is to have demonstrated the interest of defining minimal normal forces on the joints and to have presented ideas to quantify them.

The solution process presented could certainly be clarified, refined and should be implemented in a computer program and tested before the interest of the technique to solve practical problems involving real structures can be fully evaluated.

The formal treatment of the non-standard character of masonry greatly complicates the analysis of masonry structures. For many structures, the implications are so strong that it is possible to predict that no usable safe domain will be identifiable. It is then necessary to rely on engineering judgement, possibly using more detailed survey or testing and monitoring the structure.

In any case, the systematic investigation of classical hypothesis used by structural engineers to assess the safety of existing masonry constructions is extremely valuable. The limit analysis is certainly a good candidate as it is widely used and as it is based on a small set of basic assumptions.

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