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Analytical and Numerical Analysis on the Collapse Mode of Circular Masonry Arches

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Abstract In this paper, an analytical and numerical analysis on the collapse mode of circular masonry arches is presented. Specific reference is made to the so-called Couplet-Heyman problem of finding the minimum thickness necessary for equilibrium of a masonry arch subjected to its own weight (Heyman 1977). The note reports the results of an on-going research project at the University of Bergamo. First, analytical solutions are derived in the spirit of limit analysis, according to the classical three Heyman hypotheses and explicitly obtained in terms of the unknown angular position of the intrados hinge at the haunch, the minimum thickness to radius ratio and the non-dimensional horizontal thrust (Colasante 2007, Cocchetti et al. 2010). Results are then compared to Heyman solution. Though only the first of these three characteristics is perceptibly influenced in engineering terms, especially at increasing opening angle of the arch, the treatment settles an important conceptual difference on the use of the true line of thrust, along the line of Milankovitch work. Second, numerical simulations by the Discrete Element Method (DEM) in a Discontinuous Deformation Analysis (DDA) computational environment are provided, to further support the validity of the obtained solutions, with good overall matching of the obtained results (Rusconi 2008, Rizzi et al. 2010).

Keywords: Masonry arch, Couplet-Heyman problem, statics, limit analysis, collapse mechanism, DEM, DDA

Introduction

This paper addresses the classical problem of finding the minimum thickness required for equilibrium of a circular masonry arch, with general angle of embrace, subjected only to its own weight (Fig. 1). This issue is often referred-to as the *Couplet-Heyman problem* (see Heyman 1977 and the introductory review in Cocchetti et al. 2010, with references quoted therein).

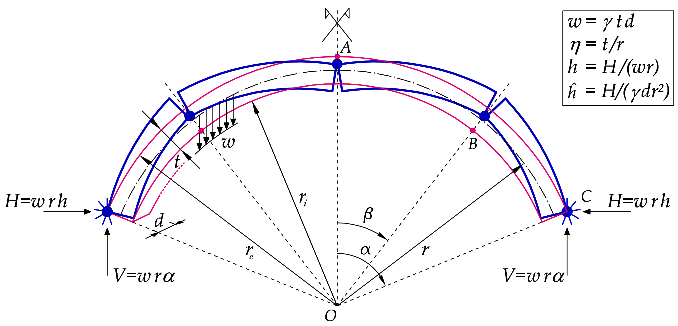

Figure 1: Sketch of a symmetric circular arch subjected only to its own weight, with all characteristic parameters and representation of its symmetric five-hinge rotational collapse mechanism

Given the value of the half-opening angle α of the arch, one attempts the determination of the following three basic parameters, which characterise the five-hinge rotational collapse mechanism: the angular position β of the haunch hinge B, measured e.g. from the vertical axis of symmetry at crown A; the minimum value $\eta=t/r$ of the thickness to radius ratio still allowing for equilibrium; the corresponding non-dimensional horizontal thrust of the arch $h=H/(wr)$, acting in such a limit state or, alternatively, the non-dimensional horizontal thrust $\hat{h}=H/(\gamma dr^2)=\eta h$ normalised as to resort just to known material (specific weight γ) and geometrical (out-of-plane depth d and radius r) parameters ($w=\gamma td$ is the specific weight per unit length of geometrical centerline of the arch).

For the solution of this problem, Heyman has provided, based on three classical behavioral assumptions (1−sliding failure does not occur; 2−masonry has an infinite compressive strength; 3−masonry has no tensile strength), useful analytical formulas which can be applied to determine the above-mentioned characteristics. While attempting re-derivation of these outcomes, slightly different results were constantly obtained (Colasante 2007), which motivated a throughout investigation on the subject (Cocchetti et al. 2010). It turns-out that Heyman has obtained his formulas by imposing the tangency condition of the resultant thrust at the haunch intrados, while such tangency condition should more correctly re-stated in terms of the true line of thrust. This originated the "CCR solution". Finally, the two solutions, which are derived by considering the self-weight as uniformly distributed along the geometrical centerline of the arch, were compared as well to Milankovitch solution, that accounts for the true location of the centers of gravity of each ideal section of the arch.

The salient results of these derivations (Cocchetti et al. 2010) are presented here, by using a compact writing of the equations that originates all three (Heyman, CCR and Milankovitch) solutions. Furthermore, the trends experienced by the solutions were also confirmed by independent DEM calculations in DDA (Rusconi 2008, Rizzi et al. 2010), as briefly reported as well in the following.

Analytical Derivation of Heyman, CCR and Milankovitch Solutions

Reference is made here to cases with entirely-general half-angle of embrace, i.e. potentially $0<\alpha<\pi$. Due to the symmetry of the problem with respect to the vertical axis at crown A, just one half AC of the arch can be considered, with only the horizontal thrust H acting on top at the crown extrados (and no shear force). At collapse, a hinge forms at the haunch intrados B ($\beta<\alpha$), which divides the half-arch into two portions AB and BC.

From the rotational equilibrium of the upper portion AB around inner hinge B and of the total half-arch AC around hinge C at the shoulder extrados, one obtains the following two equilibrium conditions in terms of the non-dimensional horizontal thrust h:

$$h = h_1 = \frac{(2-\eta)\beta sin\beta - 2(1-cos\beta)(1+\delta_M\,\eta^2/12)}{2+\eta-(2-\eta)cos\beta} \tag{1}$$

$$h = h_2 = A - \frac{2}{2+\eta}(1+\delta_M\,\eta^2/12)\ , \quad A = \alpha\, cot\frac{\alpha}{2} \tag{2}$$

where δ_M is a flag that turns-out useful to account for either Milankovitch ($\delta_M=1$) or CCR/Heyman ($\delta_M=0$) solutions. It allows to consider the true self-weight distribution in Milankovitch solution.

A third equation is thus needed to complement Eqs. (1)–(2) and to allow solving for the three unknowns β, η, h. According to Heyman indications, this relation should arise from the tangency condition of the line of thrust at the haunch intrados B. However, it appears that Heyman is rather stating this condition in terms of the resultant thrust force itself. Indeed, Heyman tangency condition reads (W_1 being the weight of the upper portion AB of the arch):

$$\frac{W_1}{H_H} = \frac{wr\beta}{wrh_H} = \frac{\beta}{h_H} = tan\,\beta \quad \rightarrow \quad h_H = \beta\, cot\,\beta = \beta\frac{cos\,\beta}{sin\,\beta} \tag{3}$$

Instead, by more correctly re-stating the tangency condition in terms of the true line of thrust, with eccentricity $e=e(\beta)$ from centerline (CCR and Milankovitch solutions), one has:

$$h = h_e = h_H - \frac{\eta}{2-\eta}\delta_{CCR}(1+\delta_M\,\eta/6) \tag{4}$$

where δ_{CCR} is another flag allowing to shift from CCR/Milankovitch ($\delta_{CCR}=1$) to Heyman ($\delta_{CCR}=0$) solutions. This equation can be derived in a straight-forward manner by imposing the stationary condition $h'(\beta)=0$, as applied to $h=h_1(\beta)$ in Eq. (1), which is equivalent to the true condition $e'(\beta)=0$. Most likely, Heyman should have assumed that, for η small, the approximation $h_e \simeq h_H$ looks reasonable in engineering terms. The couples of flags (δ_{CCR}, δ_M) should thus be conceived as follows: (0,0) for Heyman, (1,0) for CCR and (1,1) for Milankovitch solutions.

In sum, the system $\{h=h_1, h=h_2, h=h_e\}$ formed by equilibrium Eqs. (1)–(2) and tangency condition (4), with $h_H=\beta\cot\beta$ inserted in it, can be solved for the three unknowns β, η, h, at any given value of α (or of the group $A=\alpha\cot\alpha/2$; notice that the dependence on α goes always through the term A entering only in Eq. (2)). The solution of this system turns-out *"cubic"* for Milankovitch, *"quadratic"* for CCR and *"linear"* for Heyman solutions, see Cocchetti et al. (2010), where explicit analytical representations of the solutions are also provided.

Comparison of the Three Solutions Results for the three solutions at some given values of the half-opening angle α are listed in Table 1.

Table 1: Heyman, CCR and Milankovitch solutions at variable half-angle of embrace α

Solution	*Arch opening*			*CHARACTERISTIC PARAMETERS*				
	α		A	β		η	h	$\hat{h}=\eta h$
	[deg]	[rad]	[1]	[rad]	[deg]	[1]	[rad]	[1]
H				0.711924	40.7902	0.0227694	0.825056	0.0187860
CCR	60	1.04720	1.81380	0.688721	39.4608	0.0228482	0.825094	0.0188519
M				0.688718	39.4606	0.0228489	0.825052	0.0188515
H				1.02677	58.8293	0.105965	0.621113	0.0658164
CCR	90	1.57080	1.57080	0.951141	54.4963	0.107426	0.621772	0.0667947
M				0.950925	54.4840	0.107478	0.620881	0.0667311
H				1.31409	75.2920	0.314124	0.344942	0.108354
CCR	120	2.09440	1.20920	1.11714	64.0072	0.326547	0.349556	0.114146
M				1.11248	63.7402	0.327607	0.342263	0.112128
H				1.54066	88.2735	0.661358	0.0464391	0.0307129
CCR	145	2.53073	0.797935	0.981348	56.2271	0.740638	0.0681781	0.0504953
M				0.866901	49.6698	0.763995	0.0391491	0.0299097

Notice that the analytical limits of validity of the three solutions are obtained for a non-dimensional horizontal thrust that vanishes ($h=h_l=0$). Thus, the three solutions for the purely-rotational collapse mechanism hold for half-angles of embrace α that are less than about $150°$, respectively less than $\alpha_l^H=148.371°$ for Heyman, $\alpha_l^{CCR}=151.742°$ for CCR and $\alpha_l^M=148.444°$ for Milankovitch solutions (Cocchetti et al. 2010).

The outcomes of the three solutions are also reported in plot form in Fig. 2, at variable half-angle of embrace α. The inner hinge angular position $\beta(\alpha)$ is represented in Fig. 2a. Notice that, for Heyman solution, $\beta(\alpha)$ is a monotonic increasing function of α, whereas, for both CCR and Milankovitch solutions there appears a stationary point, respectively at ($\alpha_{s\beta}{}^{CCR}=127.788°$, $\beta_{s\beta}{}^{CCR}=64.6918°$) and at ($\alpha_{s\beta}{}^{M}=125.845°$, $\beta_{s\beta}{}^{M}=64.1635°$). The differences between the three solutions are here more appreciable, where one notes that CCR and Milankovitch solutions are very near up to peak, whereas Heyman solution clearly diverges from them at increasing α. Nevertheless, the differences in β do not influence in significative terms the values of the other parameters η and h. Indeed, as presented by Heyman in quoting historical contributions on the subject, these characteristics are not that sensitive with respect to approximate evaluations of the inner hinge position, which plays just an intermediate role in this sense. However, β is a true geometrical parameter of the collapse mechanism of the arch. The new solutions advanced here show, through the stationary point of $\beta(\alpha)$, that the haunch hinge first increases and then decreases back at raising α, going to zero in the limit case with $h \to h_l=0$.

The trends for $\eta(\alpha)$ and $h(\alpha)$ are depicted in Figs. 2b–2c. They are both monotonic, increasing for η (meaning that the thickness necessary for equilibrium has to increase at increasing opening angle of the arch) and decreasing for h (meaning that the horizontal thrust that the arch is able to transmit in the limit equilibrium condition decreases at increasing angle of embrace). Notice that h decreases from 1 to 0 as α grows from 0 to α_l. Accordingly, η increases (and reaches the limit value $\eta_l^{CCR}=1$ for the CCR solution). This shows that arches with $\alpha>90°$ get weaker and may stand just at the price of an increase in arch thickness. The trend for the non-dimensional horizontal thrust $\hat{h}=\eta\, h$ is reported as well in Fig. 2d. Notice that, since η and h get zero respectively on the two limits $\alpha=0$ and $\alpha=\alpha_l$, the trend of $\hat{h}(\alpha)$ is necessarily bell-shaped, also for Heyman solution, with a stationary point that appears at ($\alpha_{s\hat{h}}{}^{H}=120.918°$, $\beta_{s\hat{h}}{}^{H}=75.7771°$) for Heyman, ($\alpha_{s\hat{h}}{}^{CCR}=122.836°$, $\beta_{s\hat{h}}{}^{CCR}=64.3969°$) for CCR and ($\alpha_{s\hat{h}}{}^{M}=121.426°$, $\beta_{s\hat{h}}{}^{M}=63.9130°$) for Milankovitch solutions. This shows that there is a well-defined value of α, at around $120°$, that leads to the maximum thrust that the arch is able to sustain in the limit equilibrium condition, at given material and geometrical parameters γ and d, r, that would be assigned at design stage. Note that the $\alpha_{s\hat{h}}$ positions of the stationary points of $\hat{h}(\alpha)$ are slightly different than the $\alpha_{s\beta}$ stationary locations of $\beta(\alpha)$, which are nearer to $130°$.

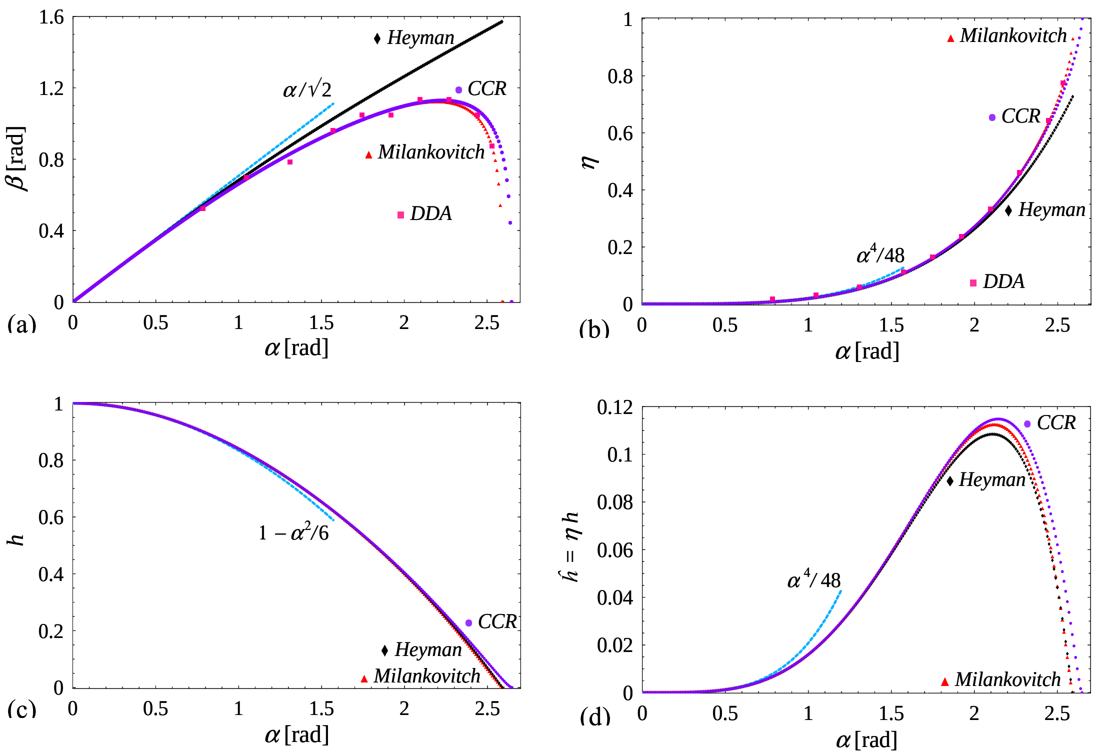

Figure 2: Heyman, CCR and Milankovitch solutions at variable half-angle of embrace α: (a) Angular position β of the haunch hinge; (b) Thickness to radius ratio η; (c) Non-dimensional horizontal thrust h; (d) Non-dimensional horizontal thrust ĥ = □ h. The trends for α small are plotted as well. Numerical DDA solutions are also scored in (a) and (b)

For both Figs. 2a–2b CCR appears in the fork made by Milankovitch and Heyman solutions, with inverted roles in the two cases (see also results in Table 1). However, for Figs. 2c–2d the roles are somehow interchanged, with Heyman between Milankovitch and CCR solutions in Fig. 2c and Milankovitch between Heyman and CCR solutions in Fig. 2d, at least until at around peak. Notice that, surprisingly, Heyman solution, which acts on h in stating the tangency condition, turns-out very accurate for the estimation of h itself and almost superimposed to Milankovitch solution, until α_l (despite clear divergence on β). Nevertheless, differences on $\hat{h}$ are still visible in Fig. 2d near peak.

Finally, in Fig. 2 the approximations of the three characteristic parameters for α small are also reported. These approximations are the same for the three solutions, that turn-out indistinguishable for small values of α. In Figs. 2a−2b the numerical results scored by the following DDA solution are superimposed as well for direct comparison purposes. The astonishing matching can be appreciated.

Numerical DEM Simulations by a DDA Formulation

A numerical analysis with a DEM formulation (*DDA for Windows*, *v. 1.6*, freely downloaded from *sourceforge.net*) has been carried-out, in view of confirming the trends experienced by the analytical solutions (Rusconi 2008, Rizzi et al. 2010).

First, four different friction coefficients $\mu=\tan\varphi$ of the joints of a full semi-circular arch ($\alpha=90°$) with sub-critical thickness, discretised with *36* blocks (*5*° voussoirs) were imposed, showing a purely sliding collapse for $\varphi=0°$, a mixed sliding-rotational mode for $\varphi=10°$ and $\varphi=20°$ and a purely rotational mechanism for $\varphi=30°$. The subsequent analyses were then run with a high value of friction angle, $\varphi=50°$, in view of complying with Heyman hypothesis−1.

Second, semi-circular arches with critical thicknesses from both Heyman ($\eta_H=0.105965$) and CCR ($\eta_{CCR}=0.107426$) solutions, made with variable number of blocks (*24*, *30*, *36*, *60*, *72*, *90*, *108*, *144*, *180*), i.e. with (*7.5*°, *6*°, *5*°, *3*°, *2.5*°, *2*°, *5/3*°=*1.67*°, *1.25*°, *1*°) voussoirs, were analysed, by investigating the position of the resulting haunch hinge β. Results in terms of β were not influenced by either $\eta=\eta_H$ or $\eta=\eta_{CCR}$ and were displaying overall a hinge position β that was definitely more on the side of CCR and Milankovitch solutions (Fig. 3).

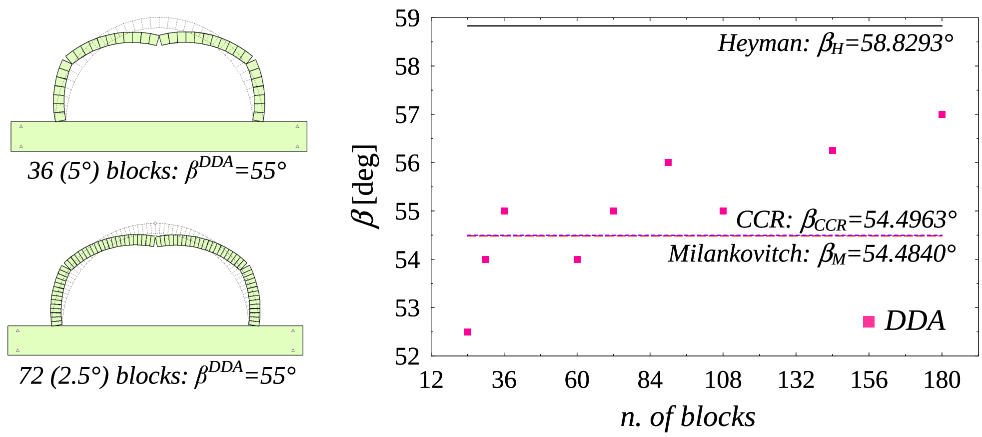

Figure 3: DEM (DDA) numerical results for a full semi-circular arch (α=90°) made with variable number of blocks, with comparison to Heyman, CCR and Milankovitch solutions. Angular position of the haunch hinge β

A third analysis was attempted to determine the critical value of η of full semi-circular arches with variable number of blocks. The obtained numerical results are shown in Table 2. Notice that the DDA solution tends to diverge, almost linearly, at increasing number of blocks, since the continuum is turning more and more into a discontinuum. However, the relative distance between DDA and CCR/M solutions appears to be always less than the distance between DDA and Heyman solutions (i.e. CCR/M are always between Heyman and DDA solutions). This agreement further validates in academic terms CCR/M vs. Heyman solutions, Milankovitch being the nearest to DDA.

Furthermore, semi-circular arches with four blocks were considered with different so-imposed β joint, at decreasing thickness, in view of determining the critical value of η at a given potential position of the haunch hinge. This also showed good agreement with all three analytical solutions.

Finally, arches with different cut-off angles α were considered, with discrete *5*° voussoirs, to determine both β and η at the first collapse instance that appears at decreasing η. The results are reported by points scored in Figs. 2a–2b, which shows a very good agreement with CCR/M analytical solutions. Especially, the true trends for $\beta(\alpha)$ in CCR and Milankovitch solutions, through a stationary point, are confirmed, as opposed to the monotonic increasing trend of Heyman solution.

Table 2: Same as Fig. 3 Critical thickness to radius ratio η.

n. of blocks	voussoir opening	DDA	HEYMAN		CCR		MILANKOVITCH	
		η_{DDA}	η_H	$\Delta\eta/\eta_H$	η_{CCR}	$\Delta\eta/\eta_{CCR}$	η_M	$\Delta\eta/\eta_M$
		[1]	[1]	[%]	[1]	[%]	[1]	[%]
12	15°	0.10884	0.105965	2.71	0.107426	1.32	0.107478	1.27
18	10°	0.11068		4.45		3.03		2.98
24	7.5°	0.11242		6.09		4.65		4.60
30	6°	0.11405		7.63		6.17		6.11
36	5°	0.11505		8.57		7.10		7.05
60	3°	0.11985		13.10		11.56		11.51
72	2.5°	0.12198		15.11		13.55		13.49
90	2°	0.12540		18.34		16.73		16.67

Conclusions

This paper presented general solutions for the classical Couplet-Heyman problem in the statics of circular masonry arches. In the spirit of Heyman studies and based on his classical hypotheses, new analytical solutions have been reported, by correctly re-stating the tangency condition in terms of the true line of thrust. Such solutions have been assessed and confirmed by parallel numerical DEM (DDA) computations. The following main conclusions can be stated:

- CCR and Milankovitch solutions are quite tight to each other and not that dissimilar from Heyman solution in engineering terms, except for the position of the haunch hinge β, which is a true characteristic of the collapse mode. However, this does not influence significantly the other parameters η, h and $\hat{h}$, especially when $\alpha \leq 90°$, as usually considered in the literature.
- Numerical DDA simulations have produced results that, specifically for the true haunch hinge position, are overall in great agreement with the new outcomes of both CCR and Milankovitch solutions, rather than with classical results by Heyman. This further supports the validity of the analytical trends reported here.

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