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# MECHANICAL BEHAVIOUR OF MASONRY PANELS UNDER IN-PLANE LOADS

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## ABSTRACT

This work examines the mechanical behaviour and bearing capacity of masonry panels subjected to vertical and horizontal loads.
We will show that by adopting the Mohr-Coulomb failure criterion and applying some simple considerations drawn from classical limit analysis, the value of the horizontal load capable of leading to panel collapse can be predicted, together with some other essential parameters characterising the failure mode itself.
Upper bound estimates of the collapse multiplier value can be easily obtained by examining the possible mechanisms of panel collapse by sliding or rotation. Lower bound estimates can instead be determined by building suitable, non-uniform statically admissible stress fields.
The estimated collapse load values obtained for the single panel are used to make some predictions about the load capacity of a masonry wall subjected to both vertical and horizontal loads.

*Keywords:* *Masonry panels, In-plane loads, Limit analysis*

## 1. INTRODUCTION

Determining the bearing capacity of a masonry wall subjected to loads acting in its midplane has been the subject of a great deal of experimental and theoretical research studies conducted on a relatively systematic basis since the 1970s (see, for example, [1, 2]). Despite such wide-ranging efforts, the true mechanisms underlying the resistance of masonry still do not seem to have been completely elucidated, thereby making this issue one of the most fundamental unresolved questions in the field of structural mechanics even today.
In the work presented herein we intend to show that is possible to evaluate the horizontal limit load of a masonry panel and the collapse mechanisms by using one single simple failure criterion for the masonry and the typical tools of limit analysis. After some introductory considerations, the following sections address the problem of determining the upper and lower bounds of the limit load of a panel, respectively obtained by examining suitable types of kinetically compatible collapse mechanisms and non-uniform statically admissible stress fields. For the sake of simplicity, we limit the study to the particular case in which the masonry is infinitely resistant to compression and entirely unable to transmit tensions. We will lastly show how the results obtained for a single panel can be used to determine an approximate limit value for the horizontal load of a masonry wall, using the example of a simple vertical rectangular wall with an opening at its base.

## 2. THE MASONRY PANEL

The stated problem is to determine the horizontal limit load of a masonry panel of rectangular shape, fixed at its base and subjected at its top to a distribution of vertical loads having a resultant of assigned magnitude, $P$, acting parallel to the panel vertical axis at a distance $e$, and a distribution of horizontal actions statically equivalent to a force, $\lambda P$, where $\lambda$ represents the horizontal force multiplier (see Fig. 1). The panel's own weight is neglected in the calculations.

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For our purposes here, the masonry panel is schematized as a two-dimensional body, in a state of plane stress, composed of a rigid-plastic material. The chosen failure criterion is the Mohr-Coulomb criterion, defined by the relation:

$$|\tau| \le c - k\sigma \, , \tag{1}$$

between the normal and tangential components, σ and τ, respectively, of the generic stress vector.

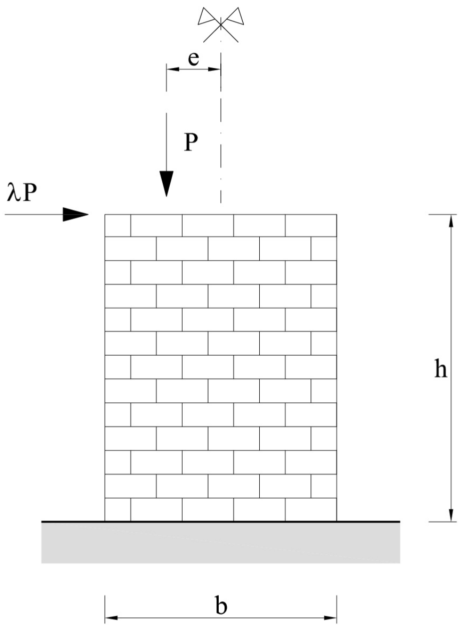


**Fig. 1** Masonry panel subjected to loads acting at its top

The cohesion *c* and the tangent of internal friction angle, *k*, are related to the masonry's tensile and compressive strengths, $\sigma_t$ and $\sigma_c$, (both assumed to be positive) according to the well-known relations:

$$c = \frac{\sqrt{\sigma_c \sigma_t}}{2}, \quad k = \tan\phi = \frac{\sigma_c - \sigma_t}{4}. \tag{2}$$

Equations (2) show that, for finite values of the tensile and compressive strengths, *c* and *k* are two limited quantities, and are both greater than zero if $\sigma_c > \sigma_t$. In the particular case in which the compression resistance is instead infinite, $k \to \infty$ and $c \to \infty$, and (1) reduces to

$$\sigma \le \sigma_t \, . \tag{3}$$

In other words, in this limit case the Mohr-Coulomb criterion ends up coinciding with Galileo's criterion.

In the following we limit the analysis to the particular case where $\sigma_c \to \infty$ and $\sigma_t = 0$. The collapse value of the horizontal force multiplier, λ, is estimated by determining the upper and lower bounds through application of the basic theorems of limit analysis. For the sake of simplicity, we assume that the masonry can be modelled as a standard rigid − perfectly plastic material. In this regard, it should be recalled that Radenkovic's theorems enable extending the static and kinematic theorems of limit analysis also to the case of non-standard materials [3].

## 3. UPPER BOUNDS OF THE LIMIT LOAD

The search for the upper bounds of the horizontal force collapse multiplier is conducted by means of the kinematic theorem of limit analysis. Radenkovic's first theorem guarantees that, even for non-standard materials, the true value of the collapse multiplier is in any event smaller than those corresponding to collapse mechanisms with associated plastic flow, which shall therefore be those taken up in the following.

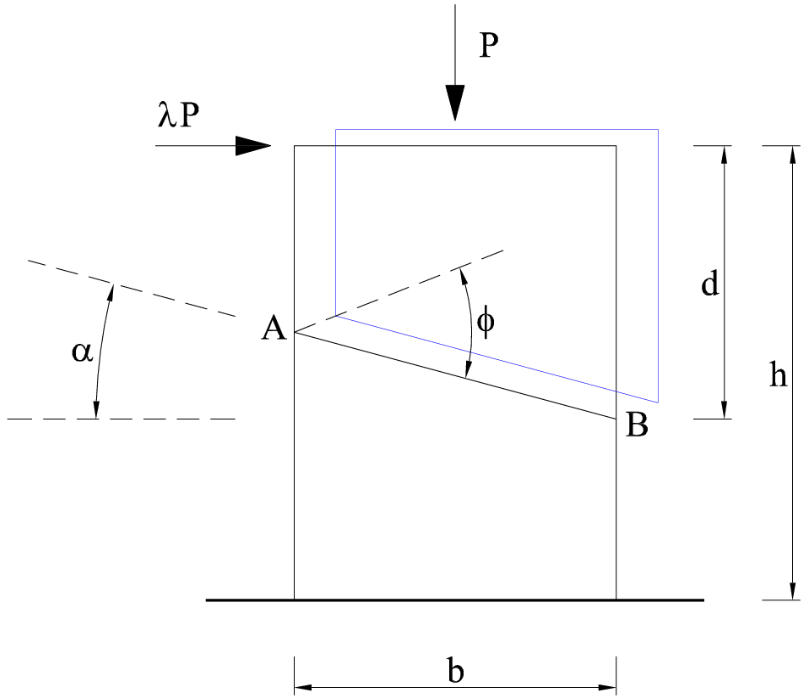


**Fig. 2** Generic collapse mechanism

Consistent with the choice of taking the Mohr-Coulomb criterion as that for masonry failure, we assume that the generic collapse mechanism is characterized by a plane surface, orthogonal to the panel midplane, along which the displacement field exhibits a discontinuity. This surface is thus identified as soon as values have been assigned, for instance, to the slope, $\alpha$, of segment AB with respect to the horizontal, and distance $d$ of point $B$ from the panel's upper base (Fig. 2). The motion that the panel's upper part undergoes with respect to the lower one is shown in the same figure. Apart from sliding one over the other, the two parts of the panel also move apart in such a way that the vector representative of the velocity of any given point of the panel's upper portion is always inclined with respect to $AB$ by the material's internal friction angle.
Along the limit surface, expression (3) reduces to simply $\sigma = 0$ and the equilibrium equations for the panel upper portion immediately enables obtaining the relations:

$$\lambda = \frac{b+2e}{2d}, \qquad \alpha = \arctan\left(\frac{1}{\lambda}\right). \tag{4}$$

For a panel of fixed dimensions, the strict upper bound of the collapse multiplier is therefore:

$$\lambda_c = \frac{1+2\xi}{2\eta}, \tag{5}$$

where $\eta = h/b$ is the slenderness of the panel, and $\xi = e/b$ is the ratio of the eccentricity of the resultant of the vertical compressive actions to the panel width.

## 4. LOWER BOUNDS OF THE LIMIT LOAD

According to the static theorem of limit analysis, the search for the lower bounds of the load multiplier can be carried out by examining a suitable collection of stress fields that are in equilibrium with the applied loads and at the same time compatible with the bounds imposed on the material's strength.
We assume that the panel is divided into a stress-free non-reacting part and a reactive portion in a state of monoaxial compression.

Since, by the assumed hypotheses, no body forces are acting on the panel, it can immediately be concluded that the isostatic lines in the panel's reactive portion are straight line segments [4].

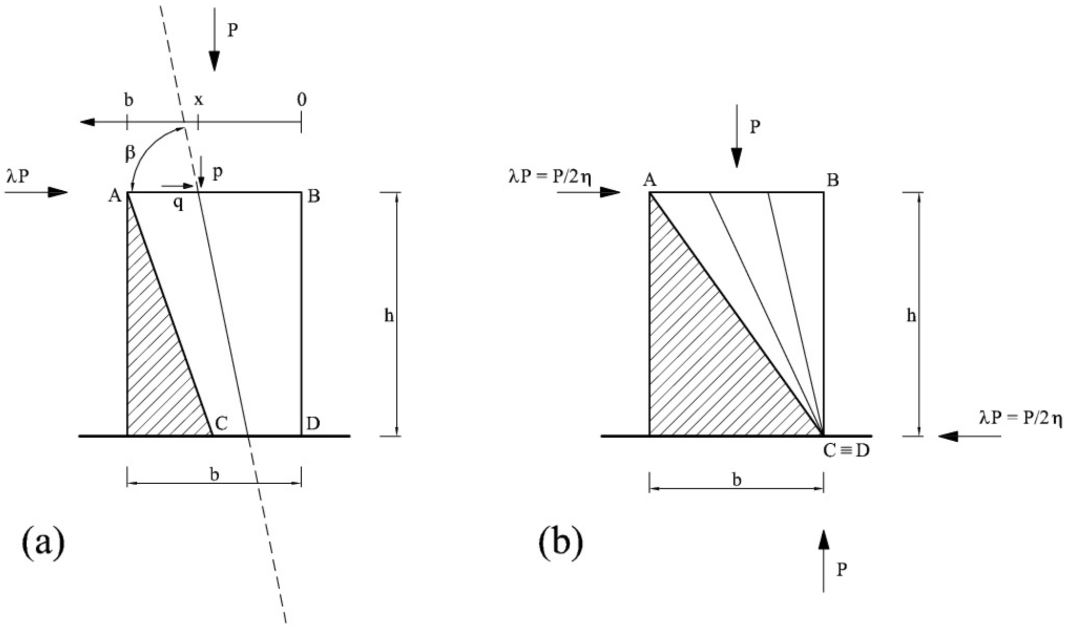


**Fig. 3** Generic isostatic line of compression in the panel's reactive part (a); plot of the isostatics for $\lambda = \lambda_s$ (b)

We denote by $\beta = \beta(x)$ the inclination with respect to the horizontal of the isostatic line of compression passing through the point situated on the abscissa $x$ of the panel's upper base (Fig. 3a). The horizontal and vertical components $q$ and $p$ of the surface forces acting on this same point are related by the equation:

$$q = \frac{p}{\tan\beta}, \tag{6}$$

and the resultant of the horizontal actions can be expressed as:

$$\lambda P = \int_0^b \frac{p(x)}{\tan\beta(x)} dx \cdot \tag{7}$$

Equation (7) shows that for an assigned distributed vertical load, $p(x)$, multiplier $\lambda$ increases with decreasing values of angle $\beta$. In this regard, bearing in mind that all the isostatics have been assumed to intersect the panel lower base, it follows that at every point on the upper base the following bound must be respected:

$$\tan\beta \geq \frac{h}{x}, \tag{8}$$

which indicates that angle β cannot fall below a minimum value.
Thus, if we set $\tan\beta = h/x$ in (7), recalling that the straight line of action of the resultant of the vertical actions is at distance $e$ from the panel vertical axis, it follows immediately that the expression for the maximum statically admissible value, $\lambda_s$, of the horizontal force multiplier for a panel of fixed dimensions can be expressed as:

$$\lambda_s = \frac{1}{P}\int_0^b \frac{xp(x)}{h} dx = \frac{1}{Ph} P\left(\frac{b}{2} + e\right) = \frac{1+2\xi}{2\eta}, \tag{9}$$

where, as usual, $\eta = h/b$ is the panel slenderness and $\xi = e/b$.

The situation corresponding to this multiplier value is represented graphically in figure 3b. The isostatics all pass through the lower base vertex $D$ and the compressions moreover diverge at $D$ [2]. The values of the horizontal force multiplier furnished by (9) are therefore statically admissible only in the limit case of masonry with infinite compressive strength. For finite values of compressive resistance, instead, the values of multiplier $\lambda_s$ that are compatible with panel equilibrium and at the same time with the masonry's strength will be consistently smaller than $\lambda_s$.

The $\lambda$ values furnished by (9) coincide with the kinetically admissible ones obtained via (5). Therefore, with sole regard to the limit case considered ($\sigma_c \to \infty$ and $\sigma_t = 0$),

$$\lambda_s = \lambda_c = \frac{1+2\xi}{2\eta}, \tag{10}$$

represents the actual value, $\lambda_p$, of the collapse multiplier. Equation (10) shows that regardless of the eccentricity of the resultant of the vertical actions, the value of $\lambda_p$ will in every case fall within the limit values $0 \le \lambda_p \le 1/\eta$.

In the case that the panel's top is free to rotate, an estimate of the value of the horizontal force multiplier can be obtained by setting $\xi = 0$ in (10), whence $\lambda_p = 1/2\eta$.

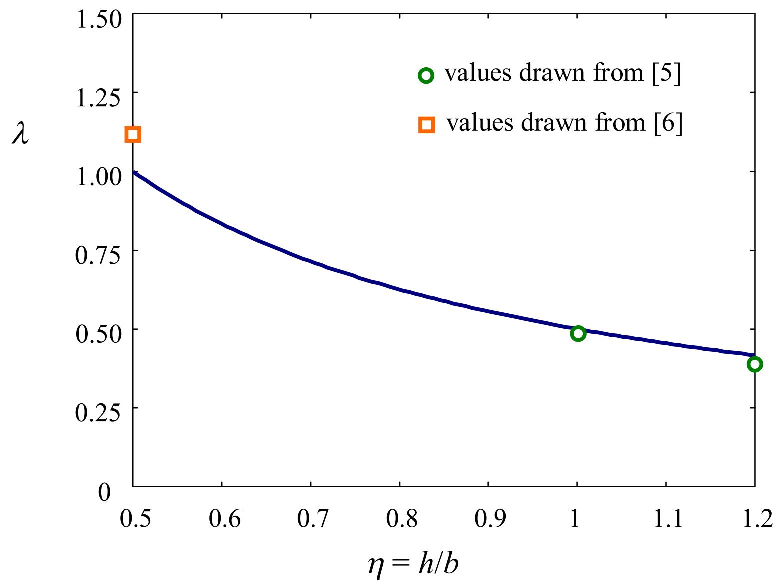


**Fig. 4** Comparison between $\lambda$ values furnished by (10) and some experimental results in the case of a centred vertical load

In this regard, a first comparison made between the values of the horizontal force multiplier furnished by Eq. (10) for the case of $\xi = 0$ and some experimental results obtained by subjecting masonry panels to vertical loads of rather modest intensity with respect to those that would cause failure by crushing (drawn from [5] and [6]), highlights the more than acceptable agreement between the two sets of values (Fig. 4).

When the panel's top is prevented from rotating, the eccentricity of force $P$ can be different from zero. In the limit case in which $P$ is applied in correspondence to the upper vertex $A$ (and hence, $e = b/2$), $\lambda_p = 1/\eta$ and the compressions are nil at all points of the panel with the exception of those belonging to the segment $AD$, where $\sigma_c \to \infty$.

## 5. AN EXAMPLE APPLICATION: LIMIT HORIZONTAL LOAD OF A MASONRY WALL

In order to exemplify an application of the results described in the foregoing with reference to a single masonry panel, let us consider a wall with an opening in its base, as illustrated in figure 5. Indicating $P$ as the weight of the wall's upper portion, $ABCD$, we aim to estimate the maximum value of the horizontal action $cP$ compatible with both the equilibrium of the different parts of the wall and the strength of its constituent masonry.

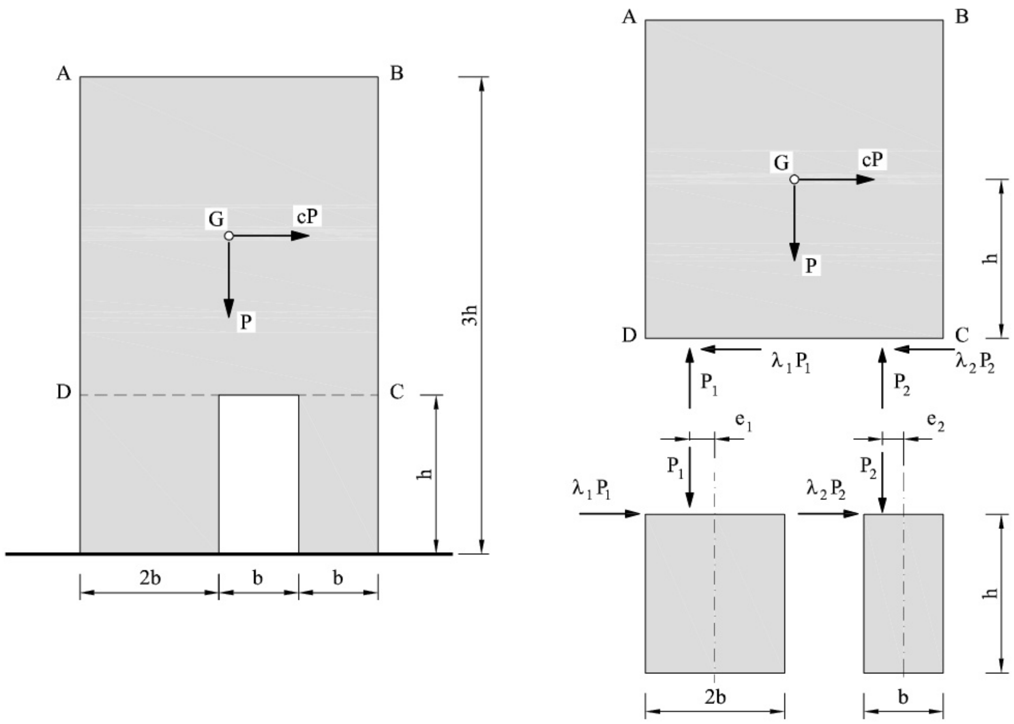


**Fig. 5** Wall with opening at its base (left); internal actions exchanged between the lower panels positioned at the wall base and the upper portion (right)

Determining the exact solution to this problem is anything but simple. Even given the simplifying hypotheses adopted regarding the masonry's resistance ($\sigma_t = 0$), the problem would not admit any solution in the event that, for instance, the masonry's specific weight were taken to be uniform and the horizontal actions uniformly distributed throughout the wall.

Here, with the aim of arriving at an approximate solution, we forsake any attempt to verify the equilibrium conditions at each point of the wall and instead choose to work through a simple scheme, that is, by imposing global equilibrium of each of the three portions into which we have divided the wall: the two panels flanking the opening on each side and the upper rectangular portion, *ABCD*, overlying the two panels.

By way of hypothesis, we assume that the two lower panels are in a limit condition. Moreover, if we neglect the panel's own weights with respect to the weight of the overlying part, by virtue of Eq. (10), we can set:

$$\lambda_1 = \frac{1+2\xi_1}{2\eta_1}, \quad \lambda_2 = \frac{1+2\xi_2}{2\eta_2}, \tag{11}$$

where $\xi_1 = e_1 / 2b$, $\xi_2 = e_2 / b$, $\eta_1 = h / 2b$ and $\eta_2 = h / b$.

By inserting (11) into the equilibrium equations of the wall portion overlying the two panels flanking the opening, we obtain the expressions for the vertical compressive actions exerted on the two base panels:

$$P_1 = P\left(\frac{1-2\xi_2}{3+4\xi_1-2\xi_2}\right), \qquad P_2 = P\left(\frac{2+4\xi_1}{3+4\xi_1-2\xi_2}\right). \tag{15}$$

Moreover, from $\lambda_1 P_1 + \lambda_2 P_2 = cP$ it follows:

$$c = \frac{2+4\xi_1}{3+4\xi_1-2\xi_2}\frac{b}{h}. \tag{16}$$

Relations (15) and (16) express the variations in the values of the resultant of the vertical actions exerted on the two base panels, $P_1$ and $P_2$, as well as in the value of the horizontal force multiplier $c$, with varying values of the two eccentricities $\xi_1 = e_1 / 2b$ and $\xi_2 = e_2 / b$.
The goal now is to determine the maximum value of multiplier $c$ under the conditions:

$$P_1 > 0 \quad \text{e} \quad P_2 > 0 . \tag{17}$$

Conditions (17) always hold for $-1/2 \le \xi_1 \le 1/2$ and $-1/2 \le \xi_2 \le 1/2$. By observing that $c \le b/h$, as it is also confirmed by the plot of the function $c = c(\xi_1, \xi_2)$ shown in figure 6a, it can be concluded that $c$ reaches its maximum value in correspondence to $\xi_2 = 1/2$ (with any $\xi_1$ whatever), where it is

$$c_{\max} = \frac{b}{h} . \tag{18}$$

Therefore, when the horizontal force is direct towards the right, as shown in figure 5, the panel to the left of the opening is unloaded ( $P_1 = 0$ ), while the right-side panel is subjected to the action of a force $P_2 = P$ acting in correspondence to the left vertex of the upper base. If the upper part of the wall is assumed to be infinitely resistant, this situation corresponds to the actual collapse mechanism for the masonry wall.

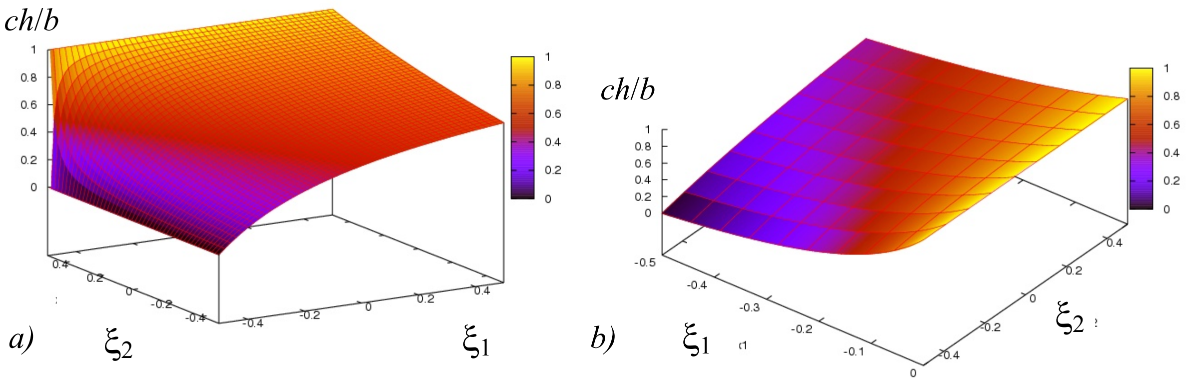


**Fig. 6** 3D plot of the values of the horizontal force multiplier for rightward (*a*) and leftward (*b*) horizontal actions

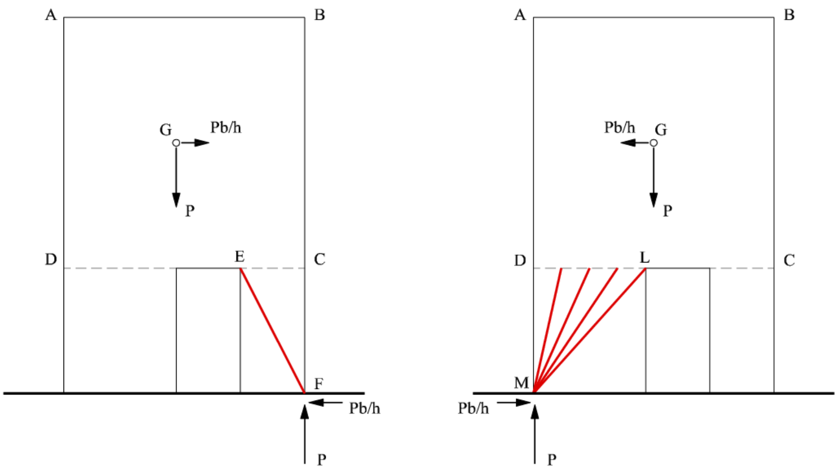


**Fig. 7** Schematic illustration of the isostatic lines of compression in the reactive zones of the base panels

If we instead consider an horizontal action pointing in the direction opposite that shown in figure 5, calculations analogous to the foregoing enable concluding that $c_{\max} = b/h$ (Fig. 6b), $P_1 = P$ and

$P_2 = 0$. In this case, however, it can been seen that positive values of $\xi_1$ are unacceptable in that they would involve vertical tensile actions on one of the two panels. Moreover, in the left panel, which sustains the weight of the entire upper part, the resultant of the vertical actions turns out to have an eccentricity $\xi_1 = 0$. In other words, the compressed panel manages to develop only half that which would be its maximum bearing capacity for horizontal actions. Once again, if the upper part of the wall is assumed to be infinitely resistant, this situation corresponds to the actual collapse mechanism for the masonry wall.

## 6. CONCLUSIONS

The mechanical behaviour and bearing capacity of masonry panels subjected to vertical and horizontal loads have been investigated by adopting the Mohr-Coulomb failure criterion and applying some simple considerations drawn from classical limit analysis. Under these hypotheses, the value of the horizontal load capable of leading to panel collapse have been predicted, together with some other essential parameters characterising the failure mode itself.

Upper bound estimates of the collapse multiplier value have been obtained by examining the possible mechanisms of panel collapse − by sliding or rotation. Lower bound estimates are instead arrived at by building suitable, non-uniform statically admissible stress fields.

The results obtained with reference to a single masonry panel are then used to estimate the value of the limit load of a simple masonry wall subjected to both vertical and horizontal loads. This example application shows that the proposed method can also be extended to cases of walls with more complicated shapes, for example, having one or more openings.

## REFERENCES


[1] Magenes G., Calvi G.M. (1997). In-plane seismic response of brick masonry walls. *Earthquake Engineering and Structural Dynamics*, 26, 1091-1112.

[2] Roca P. (2006). Assessment of masonry shear-walls by simple equilibrium models. *Construction and Building Materials*, 20, 229-238.

[3] Lubliner J. (1998) *Plasticity theory.* Macmillan Pub Co.

[4] Mansfield E.H. (1968). Tension Field Theory. In: *Proc. 12th Int. Congress on Applied Mechanics*, 305-320, Springer, New York.

[5] Abrams D., Shah N. (1992). Cycling load testing of unreinforced masonry walls. Techn. Report #92-26-10, Adv. Const. Techn. Center Project, Dept. of Civil Engineering, University of Illinois.

[6] Lourenco P.B., Oliveira D.V., Roca P., Orduna A. (2005). Dry joint stone masonry walls subjected to in plane combined load. *Journal of Structural Engineering*, 131 (11), 1665-1672.