ON THE CONTRIBUTION OF BERNARDO VITTONE IN THE DESIGN OF MASONRY DOMES

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ABSTRACT

Masonry domes can be considered among the oldest structures built by humans. In this paper the study of architectural and structural outline of domes has been limited to a particular historical period, the XVIII century. During this period analytical approaches have been developed concurrently to the consolidation of the proportional methods, characterized by the publication of architectural treatises in which several graphical constructions for the design of masonry domes can be found. It’s well known that Carlo Fontana, in 1694, was the first to show in detail the geometrical rules for the design of a masonry dome. Later the Italian architect Bernardo Antonio Vittone (1704-1770), in his treatise *Istruzioni elementari dell’architettura civile* of 1760, proposed some geometrical variations to the rules written by Fontana to increase the height of the dome and the slope of the meridian section at the intersection with the lantern.

In this paper the variations proposed by Vittone has been analyzed through both the analytical and numerical approaches. In the first case the membrane theory for shells of revolution with axisymmetric loads has been used; in the latter case a finite element model with axisymmetric formulation has been performed. The results confirm that the dome profile introduced by Vittone has not only architectural advantages, but also structural, above all in terms of hoop tensile stresses.

Keywords: Masonry domes, Membrane theory, Bernardo Vittone

1. INTRODUCTION

The design of masonry domes over the centuries has gone through several methods due to the growth of the human knowledge and the technology development. Until the XVIII century, the architects approached the design of such structures by means of proportional principia and geometrical rules of construction mainly derived by the experience.

Leaving out the architectural treatises of the XV and XVI centuries, in which some instructions on the dome’s profile can be found – as in [1] – the first and most famous treatise that shows in detail the geometrical rules for the construction of a masonry dome was written by Carlo Fontana (1638-1714) in 1694, *Il Tempio Vaticano e sua Origine* [2]. During the XVIII century the study of masonry domes becomes subject for an heated debate between the mathematical schools and the professional architects, because the former provided too theoretical solutions for the needs of the sites.

Bernardo Vittone (1704-1770) was an Italian architect, whose principal works can be observed in the Piedmont, belonging to the second group before mentioned and, in his treatise *Istruzioni elementari dell’architettura civile* of 1760 [3], continued to make proposals of geometrical dimensioning for the design of structures. In particular, he proposed some geometrical variations to the rules written by Fontana about masonry domes explained in two different models, in order to increase the height of the dome and the slope of the meridian section at the intersection with the lantern.

Till today, this debate is the subject of numerous commentaries and discussions, in which the thought of Bernardo Vittone still finds agreements [4] and criticisms [5].

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The aim of this work is to demonstrate that the variations proposed by Vittone in the second model not only produce architectural benefit to the entire complex but also structural advantages in terms of tensile hoop stresses. These characteristics have been investigated through the membrane theory and by means of numerical analysis using finite elements with axisymmetric formulation.

2. BERNARDO VITTONE AND THE RULES OF CONSTRUCTION FOR MASONRY DOMES

2.1. The structural concepts in the domes of Bernardo Vittone

The static conception of Bernardo Vittone is related to the considerable expertise developed in building sites during the years. His architectural and engineering knowledge are inserted in a complex scientific and cultural framework about the studies on the design of arches and vault structures, that in the XVIII century has increased due to the new analytical approaches applied to the mechanical of structures introduced by the French and Italian mathematical schools [6]. In the context of masonry domes, the main contributions aiming at the research of the optimal shape and/or thickness and at the understanding of the structural behaviour, are due to the works of Bouguer [7], Bossut [8], Mascheroni [9], Ventuoli [10].

These new analytical tendencies in the design of structures clashed with the well-established methods based on proportional and geometrical rules, that allow to size up structural elements regardless the loads. Bernardo Vittone focused his studies and practical operations on geometrical approaches, with reference to the rules of constructions given in the most famous architectural treatises of the time, such as those of Belidor, Dérand and others [4], and then consolidated by the building practice.

In the design and construction of masonry domes Vittone is both conservative and original. The conservative approach can be observed in the conception and understanding of the structural response, which led him to use the so called “artificiosi ritegni” (artificial restraints) [3] to prevent the deformations expected for the thrust of vaulted structures. Even if the main thought about this aspect at that time, shared by many architects, was that the well-designed structures should stand without further apparatus, in addition to employ traditional solutions to assure the stability of domes, such as the spandrels at the extrados or the construction of a tiburio, Vittone often makes use of tie rods or tension rings, as in the dome of San Michele in Rivarolo [4].
At the same time some architectural choices, and inevitably also structural, seem to be very innovative, such as for the construction of three vaulted domes and the making of holes in the inner vaults for the introduction of the light (e.g. Church of Santa Chiara, Bra, Cuneo (Fig. 1a) and Santuario della Visitazione di Maria al Vallinotto, Carignano, Torino (Fig. 1b), but especially for the research of an optimal shape for masonry domes in an ellipsoid of revolution.

For the design of masonry domes Vittone looks on the works of Carlo Fontana [2], in which the rules of construction for a simple dome are defined. In the treatise *Istruzioni elementari dell’architettura civile* [3] Vittone starts from the rules of Fontana to propose two graphical constructions in order to solve some architectural and static questions. In the latter model, more complex than the first, the tendency to obtain an elliptical profile constituted by a series of circular arches is explicit.

In this way the architect obtains a dome higher than that of Fontana, for the same inner diameter, and a better structural response in terms of hoop stresses.

### 2.2. The second model of rules of construction for masonry domes

In this section the rules of construction of the second model proposed by Vittone for the geometrical design of masonry domes are shown. In Fig. 2a-2b the profiles obtained by the application of the rules of Fontana and Vittone respectively are represented. In this work the architectural, and then mechanical, contribution of the lantern is omitted.

![Fig. 2](image-url)

**Fig. 2** Rules of construction for simple domes proposed by Fontana (a) and Vittone (b)

![Fig. 3](image-url)

**Fig. 3** Element of a shell of revolution with axisymmetric loads

Starting from the pier AB, that is the same of the Fontana’s model, Vittone makes further higher the impost of the dome, from which the intrados and the extrados lines of the vault stand out. Each curve consists of three circular arches joined so as to keep constant the value of the tangent slope. Each of the six circular arches is characterized by its own centre and radius of curvature which are different from each other, so that the graphic construction appears fairly complex. The main intent is to obtain...
a shape for the meridian section as close to the elliptical one by means of circular arches in order to simplify the realization of the centers. The difference between the two profiles starts at the beginning: in spite of a lower level of the vault impost, it should be noted that the first radius of curvature in the profile proposed by Vittone is significantly greater than the other (Fig. 2). In this way an higher dome is obtained with a shape able to better withstand the gravity loads and the resulting horizontal forces.

**3. ANALYTICAL APPROACH: APPLICATION OF THE MEMBRANE THEORY**

**3.1. Stresses in shells of revolution with axisymmetric loads: the case of ogival domes**

The membrane theory is generally applied to shell structures considered to be “thin”, which are characterized by a thickness less than about 5 per cent of the meridian radius of curvature [11].

Let us consider a shell of revolution, with constant thickness and subjected only to its own weight, placed in a reference system shown in Fig. 3. In this case the stress distribution is independent of \( \theta \), and the governing equilibrium equations are

\[
\frac{d}{d\phi} (r N_{\phi}) - r_1 N_{\phi} \cos \phi = -p_r r_1
\]

\[
\frac{N_{\phi}}{r_1} + \frac{N_{\phi}}{r_2} = p_r
\]

where

\[
\begin{align*}
   r &= r_2 \sin \phi \\
   p_{\phi} &= p \sin \phi \\
   p_r &= -p \cos \phi
\end{align*}
\]

in which \( r_1 \) and \( r_2 \) are the radii of curvature, as shown in Fig. 3, and \( p \) is the weight per unit of area of the shell.

Following the theory developed by Flügge [12], solving (2) for \( N_{\phi} \) and substituting the result into (1), a first-order differential equation for \( N_{\phi} \) is obtained

\[
\frac{d(r N_{\phi})}{d\phi} \sin \phi - r N_{\phi} \cos \phi = r R q_r \cos \phi \sin \phi - r_1 r_2 p_{\phi} \sin^2 \phi
\]

so that, combining the two terms at the left-hand side to form a total derivative and considering the first relation of (3), \( N_{\phi} \) may be found by an integration

\[
N_{\phi} = \frac{1}{r_2 \sin^2 \phi} \left[ r_r (p_r \cos \phi - p_{\phi} \sin \phi) \sin \phi d\phi + C \right].
\]

The equation (5) can be interpreted as an equilibrium condition for the shell cut by a parallel circle with \( \phi = \text{const} \), in which the constant \( C \) represents the effects of loads above a predefined circle \( \phi = \phi_0 \). The hoop stresses \( N_{\phi} \) follow from (2).

If the shape of the meridian is a quarter of a circle, after the revolution process, an hemispherical dome is obtained and, under only the action of its own weight, it’s well known by the literature that the meridian stresses are anywhere of compression, while the hoop stresses, of compression at the top, became of tension after a value of \( \phi = 51.82^\circ \).

If the meridian meets the axis of revolution with a value of the slope lower than a right angle, a pointed dome results, characterized by an apex at the top. In this case the state of stress is obviously dependent by the angle \( \phi_0 \) at which the intersection of the meridian arch with the axis of revolution occurs. Owing to the variability of the radius of transversal curvature
\[ r_2 = \frac{r}{\sin \phi} = a \left( 1 - \frac{\sin \phi}{\sin \phi} \right), \]  

where \( a \) is the constant radius of curvature \( r_1 \) of the meridian, the equation (5) becomes

\[ N_{\phi} = -pa \left( \cos \phi_0 - \cos \phi_ \right) \left( \phi - \phi_0 \right) \sin \phi_0 \]  

\[ \left( \sin \phi - \sin \phi_0 \right) \sin \phi \]  

(7)

![Diagram](image_url)

**Fig. 4** Force diagrams for the ogival shape proposed by Fontana. \( \phi_0 = 8.21^\circ \)

and the hoop stresses \( N_\theta \) are given by

\[ N_\theta = -\frac{pa}{\sin^2 \phi} \left[ \left( \phi - \phi_0 \right) \sin \phi_0 - \left( \cos \phi_0 - \cos \phi \right) + \left( \sin \phi - \sin \phi_0 \right) \cos \phi \sin \phi \right]. \]  

(8)

In the following parametric descriptions the membrane stresses will be expressed in function of the internal diameter of the dome \( d_{int} \), in order to compare several types of domes. The general section of a “simple dome” obtained by the graphical construction shown by Fontana is ogival in shape, with a value of \( \phi_0 \) of about 8.21° and \( a = 0.583 \ d_{int} \). In figure 4 the force diagrams obtained in a dome generated by the rules of Fontana are shown. The variations proposed by Vittone in the second method lead to an ogival dome composed by three circular arches, \( AB - BC - CD \), suitably connected along the length of the meridian, in order to have variations of the angle \( \phi \) without discontinuities (Fig. 5). The graphical rules identify the parameters of Tab. 1.

**Table 1** Geometrical parameters for the shape proposed by Vittone

<table>
<thead>
<tr>
<th>( \phi_1 )</th>
<th>( \phi_2 )</th>
<th>( \phi_3 )</th>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>( a_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.58°</td>
<td>53.13°</td>
<td>77.90°</td>
<td>0.56d_{int}</td>
<td>0.64d_{int}</td>
<td>d_{int}</td>
</tr>
</tbody>
</table>

In any circular arch, in which \( \phi_i \leq \phi \leq \phi_{i+1} \) with \( i=1,2,3 \), the value of the radius of revolution \( r \) and the radius of transversal curvature \( r_2 \) can be obtained respectively by the relations

\[ r(\phi) = r(\phi_i) + a_i \left( \sin \phi - \sin \phi_i \right) \]  

(9)

\[ r_2(\phi) = \frac{r(\phi)}{\sin \phi} = \frac{r(\phi_i)}{\sin \phi_i} + a_i \left( 1 - \frac{\sin \phi}{\sin \phi_i} \right) \]  

(10)

The membrane forces in meridian direction are given in each interval \( \phi_i \leq \phi \leq \phi_{i+1} \) by the equation

\[ N_\phi(\phi) = N_\phi(\phi_i) - \frac{pa_i}{r_2(\phi) \sin \phi} \int_{\phi}^{\phi} r_2(\xi) \sin \xi \, d\xi \]  

(11)

and the hoop stresses \( N_\theta \) follow from (2). In the formulation it is implied that \( r(\phi_i) = 0 \) and \( \phi_3 \) is the right angle. In figure 5 are shown the diagrams of the membrane forces in function of the internal
diameter and the weight, in which some discontinuities of the hoop stresses appear owing to the indeterminateness of the radius of curvature in the junction of the circular arches. Such discontinuities are balanced by the arising of local bending stresses that are not taken into account in this work. The comparison with the Fontana’s model (Fig. 4) highlights a reduction of about 25% of the tensile hoop stresses at the base of the dome, confirming that the contribution of Bernardo Vitone gave also structural, other than architectural, advantages.

![Fig. 5 Force diagrams for the shape proposed by Vitone](image)

### 3.2. Elliptical domes

With the introduction of his rules of construction, Vitone realizes the structural concept for the design of masonry domes that other famous architects sensed before him. The purpose is to obtain a dome having the shape of an ellipsoid of revolution, where the minor axis of the ellipse coincides with the diameter of the dome.

![Fig. 6 Elliptical profile fitted in the graphical construction proposed by Vitone](image)

![Fig. 7 Force diagrams in an elliptical fitted to the shape proposed by Vitone](image)
In the *Istruzioni elementari* Vittone wrote that the meridian curves, drawn through his graphical rules both for the internal and external profile, «can be regarded as elliptical arches» [3], in which, however, the respective minor axes do not properly coincide with the corresponding internal and external diameter of the dome. In this case the axis of revolution intersects the meridian at a point with non-zero tangency, so that the solid results as an elliptical pointed shell.

As previously introduced, in this work a constant thickness along the development of the vault is assumed. In order to compare the membrane forces obtained in an elliptical dome with the results of the previous section, the semi-ellipse that fits the internal profile of the construction proposed by Vittone has been considered (Fig. 6).

In an ellipsoid of revolution the radii of curvature are defined by

\[
 r_1 = \frac{a^2 b^2}{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{1/2}}, \quad r_2 = \frac{a^2}{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{1/2}}
\]

so that the equation (5) becomes

\[
 N_\phi = -p a^2 b^2 \frac{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{1/2}}{\sin^2 \phi} \int_0^\pi \frac{\sin \vec{\xi}}{(a^2 \sin^2 \vec{\xi} + b^2 \cos^2 \vec{\xi})^{1/2}} d\vec{\xi}
\]

where \(a\) and \(b\) are the semi-minor and semi-major axis of the ellipse respectively, with \(b/a \approx 1.33\). By numerically solving the integral of the equation (13) the force diagrams of the membrane stresses for the specific profile are shown in figure 7. The results are in a good agreement with the contents of figure 5.

### 4. NUMERICAL APPROACH: FEM ANALYSIS

Numerical models of the meridian section for the profiles proposed by Fontana and Vittone have been performed using 4-nodes axisymmetric finite elements in linear elastic field. Meridian and hoop stresses obtained by the only action of gravity loads have been compared also varying the restraint level at the base of the dome: in a first case only the vertical displacements have been prevented, in a second case also the radial deformation has been avoided in order to reproduce the constraint effect of the drum.

In tab. 2 the maximum values of hoop tensile stresses obtained in domes with \(d_{int} = 15\) m and specific gravity of 18 kN/m³ are reported. In figure 8 the numerical results obtained for the profiles of Fontana and Vittone are compared in terms of stress states. The results highlight that the proposal of Vittone reduces the peak value of hoop tensile stresses for the both restraint levels: in the case without radial constraint more than 30%; in the case with radial constraints the reduction is of about 20%.

<table>
<thead>
<tr>
<th>Model without radial constraint</th>
<th>Fontana’s dome</th>
<th>Vittone’s dome</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.167</td>
<td>0.108</td>
<td></td>
</tr>
<tr>
<td>Model with radial constraint</td>
<td>0.044</td>
<td>0.036</td>
</tr>
</tbody>
</table>

### 5. CONCLUSIONS

In this paper the contribution of the architect Bernardo Vittone in the design of masonry domes has been achieved by means of analytical and numerical approaches: for the first method the membrane theory has been used and for the latter finite element models have been performed. In particular the profile of a simple dome, obtained via graphical construction, proposed by Vittone has been compared with the profile previously proposed by Fontana in order to investigate the presence of structural advantages in a profile proposed by only graphical considerations. The main result is show that in the dome obtained by the rules of construction proposed by Vittone the maximum value of the tensile hoop stresses is reduced of about 20%-30% in comparison with that of Fontana.
Fig. 8 Maps of hoop stresses obtained by FEM analysis of dome profiles proposed by Fontana (a-b) and Vittone (c-d) without and with radial constraint respectively.

REFERENCES