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# LIMIT ANALYSIS OF CONFINED MASONRY SHEAR WALLS BY RIGID BLOCK MODELING WITH CRACKING UNITS

*Portioli F.*[1] , *Cascini L.*[2] , *Casapulla C.*[3], *D'Aniello M.*[4] *& Landolfo R.*[5]

## ABSTRACT

In this paper, an application of rigid block limit analysis to confined masonry shear walls is presented. A simplified micro-modelling approach is adopted, based on the discretization of the single masonry unit into two blocks separated by cohesive contacts. Failure modes, which may occur at joint interfaces as well as in the masonry unit itself, involve crushing, cracking and sliding. An iterative solution procedure of linear programming problems was used to take into account non-associative flow rule as well as brittle behaviour of cohesive joints in cracking failure.
To show the accuracy of the proposed modelling approach, a case study from the literature of a confined masonry panel is analysed. The obtained results are compared with experimental tests and with the outcomes of other modelling approaches used in the literature.

*Keywords:* *Confined shear walls, Limit analysis, Iterative linear programming, Rigid block, Cracking units*

## 1. INTRODUCTION

Rigid block limit analysis by mathematical programming of confined shear masonry walls involves different modelling issues which might affect remarkably the response and produce incorrect results if not properly taken into account [1].
A first issue is relevant to the modelling of actual boundary conditions, which generally involve the problem of restraining the vertical and rotational displacements at the top of the confined panel. It is clear that the type of restrain influences the value of the final vertical load and bending moment at the top and, consequently, the distribution of normal and shear stresses at collapse. This is due to the combined effects of elastic and plastic deformation at collapse. Another important issue for the reliable prediction of the structural response of this type of masonry shear walls is actually concerned with the possibility to include in the modelling failure of the units, which usually can be subjected to cracking in experimental tests.
Following these considerations, in this paper a rigid block model with cracking units is developed for the limit analysis of confined shear walls. A simplified micro-modelling approach is adopted, considering the masonry texture as an assemblage of rigid blocks interacting through joint interfaces. In order to consider the possible cracking of elements, masonry units are divided into two blocks. Failure modes at joint and unit interface involve crushing, cracking as well as sliding. Non associative flow rule is considered for friction behaviour and brittle behaviour of cohesive joints in tension failure is taken into account by means of an iterative solution procedure of linear programming problems. Direct formulations of the boundary conditions are developed to take into account the actual stress distribution at collapse in case of confined panels.

[1] Assistant Professor, University of Naples "Federico II", fportiol@unina.it
[2] Researcher, University of Naples "Federico II", lucrezia.cascini@unina.it
[3] Assistant Professor, University of Naples "Federico II", casacla@unina.it
[4] Assistant Professor, University of Naples "Federico II", mdaniel@unina.it
[5] Full Professor, University of Naples "Federico II", landolfo@unina.it

The modelling approach is validated against a case study from the literature of confined masonry panels. The obtained results of the analysis are compared with experimental tests and with the outcomes of other classic modelling approaches used in the literature for the investigated panels.

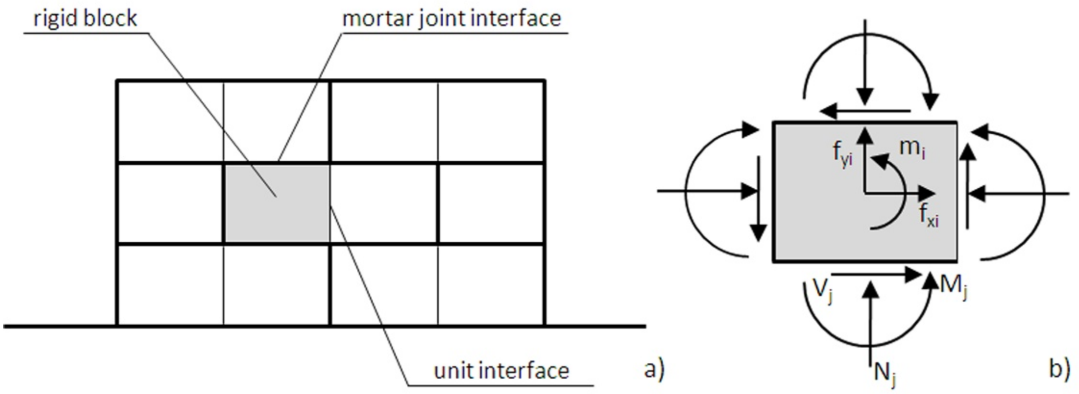


**Fig. 1** a) Proposed rigid block model with cracking of the units; b) Block and contact static variables

## 2. THE RIGID BLOCK LIMIT ANALYSIS MODEL

In this section, the modelling approach adopted for the formulation of the limit analysis problem as a mathematical program is presented and discussed in detail. Static and kinematic variables as well as equilibrium equations, constitutive laws and flow rules are expressed in matrix form.
The formulation of governing equations, which mainly refers to the work by Ferris & Tin Loi [2], has been developed for two-dimensional block assemblages.
Details of the solution procedure based on iterative linear programming problems with variable contact properties are also provided.

### 2.1. Static and kinematic variables

The structural model consists of an assemblage of rigid blocks i connected by interfaces j. The masonry unit is represented by two rigid blocks separated by a vertical contact passing through the centroid of the element itself (Figure 1a).
The static variables are represented by the internal forces (Vj, Nj, Mj ) acting at the interface j and referred to the centre of the contact (Figure 1b). These variables are collected in the vector xj.

$$\mathbf{x_j} = \begin{bmatrix} V_j & N_j & M_j \end{bmatrix}^T \tag{1}$$

The corresponding kinematic variables are the relative displacement rates at the interfaces (Figure 2). These variables are collected in the vector $\mathbf{q_j}$:

$$\mathbf{q_j} = \begin{bmatrix} \gamma_j & \varepsilon_j & \omega_j \end{bmatrix}^T \tag{2}$$

The loads are applied to the centroid of the rigid block i and are indicated with the vector $\mathbf{f_i}$:

$$\mathbf{f_i} = \begin{bmatrix} f_{xi} & f_{yi} & m_i \end{bmatrix}^T . \tag{3}$$

The loads $f_i$ are expressed as the sum of the known dead loads $\mathbf{f_{Di}}$ and live loads $\mathbf{f_{Li}}$ amplified by an unknown scalar multiplier α.

$$\mathbf{f_i} = \mathbf{f_{Di}} + \alpha \mathbf{f_{Li}} \tag{4}$$

The displacement rates at the centroid of the block *i*, that are work conjugated to the nodal loads $\mathbf{f_i}$, are collected in the vector $\mathbf{u_i}$:

$$\mathbf{u_i} = \begin{bmatrix} u_{xi} & u_{yi} & u_{\theta i} \end{bmatrix}^T \tag{5}$$

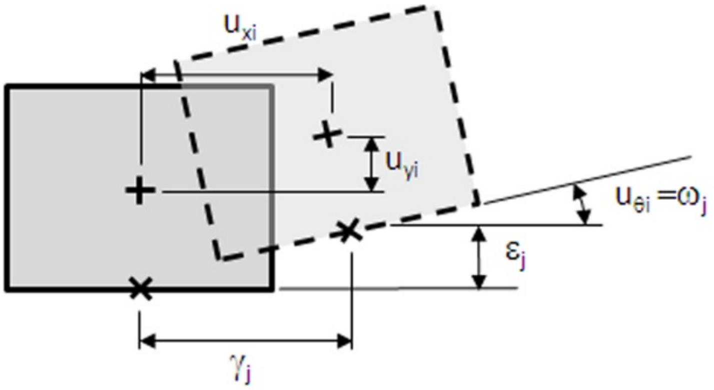


**Fig. 2** Block and contact kinematic variables

### 2.2. Equilibrium equations

For block i and contact j, the equilibrium equations are expressed in the matrix form:

$$\mathbf{A}_{\mathbf{i/j}}\mathbf{x}_{\mathbf{j}} = \mathbf{f}_{\mathbf{i}} \tag{6}$$

where $\mathbf{A}_{\mathbf{i/j}}$ is a $(3 \times 3)$ equilibrium matrix.
For the entire structure the equilibrium in the matrix form gives:

$$\mathbf{A}_{3b\times3c}\mathbf{x}_{3c} = \mathbf{f}_{3b} \tag{7}$$

being b the number of blocks and c the number of contacts, and it is obtained through assembly of matrices for each block.

### 2.3. Yield conditions

The behaviour at contact interfaces is governed by yield functions which take into account sliding and crushing failure modes. Although different contact properties can be assigned to unit and joint interfaces, yield conditions can be formulated in general terms with respect to a contact j as described in the following.
Sliding is governed by a Coulomb type criterion and is expressed by the following relationships:

$$y_j^{s\pm} = \pm\cos\phi_j V_j - \sin\phi_j N_j \le r_{j1} \tag{8}$$

where $y_j^{s\pm}$ are the yield functions for positive and negative sliding, $\phi_j$ is the friction angle and $r_{j1} = c_j \cdot \cos\phi_j$, being $c_j$ the cohesion at contact *j*.

The friction coefficient is defined as $\mu_j = \tan\phi_j$.

For crushing, a perfectly plastic behaviour in compression and tension was considered. As it is well known, in this case a parabolic yield function is obtained in the M, N domain. To reduce the resulting optimization problem to a linear program, the corresponding non-linear yield condition was approximated by eight hyperplanes. In this case, the generic expression of the admissibility condition for the internal static variables M and N at joint j is:

$$y_j^{ck\pm} = \pm\cos\psi_{jk} M_j - \sin\psi_{jk} N_j \le r_{jk} \quad \text{for } k=2,\ldots,5 \tag{9}$$

In Eq.(9) $\cos\psi_{jk}$ and $\sin\psi_{jk}$ represent the direction cosines of the normal to hyperplane *k* and $r_{jk}$ is the orthogonal distance from origin to the limit plane.

As for failure loads in tension and compression, it results $N_{c,j} = f_{cef,j} \cdot A_j$ and $N_{t,j} = f_{tef,j} \cdot A_j$, being $A_j$ the contact area at interface *j* and $f_{cef,j}$ $f_{tef,j}$ the effective values of compressive and tensile strength.

The effective values of plastic strength are determined according to Eq. (10-13), following the approach used by Orduña [3] and originally proposed for concrete limit analysis by Nielsen [4]:

$$f_{cef,j} = \nu_{c,j} \cdot f_{c,j} \tag{10}$$

$$\nu_{c,j} = 0.7 - \frac{f_{c,j}}{200} \tag{11}$$

and

$$f_{tef,j} = \nu_{t,j} \cdot f_{t,j} \tag{12}$$

$$\nu_{t,j} = 0.6 \tag{13}$$

where $f_{c,j}$ and $f_{t,j}$ are the compressive and tensile strength at contact interface $j$, expressed in MPa.
The value of cohesion $c_j$ in (8) is generally expressed as a function of effective tensile strength.
The effectiveness factors $\nu_{c,j}$ and $\nu_{t,j}$ are used to take into account the differences between rigid plastic strength assumed in limit analysis and the softening behaviour of material.
In matrix notation, the previous limit conditions (8) and (9) at a contact interface j can be written as:

$$\mathbf{N}_j^T \cdot \mathbf{x}_j \le \mathbf{r}_j \tag{14}$$

being $\mathbf{N}_j^T$ the yield function matrix and $\mathbf{r}_j$ a constant vector.
The extension of previous expression to the whole structure is straightforward.

### 2.4. Flow rule

Flow rule provides relationships between generalized strains and resultant strain rates, that is plastic multipliers in limit analysis, and is governed by the following equations:

$$\mathbf{q} = \mathbf{V}\mathbf{z} \tag{15}$$

where $\mathbf{V}$ is the flow rule matrix and $\mathbf{z}$ is the vector of plastic multipliers, being $\mathbf{z} \ge \mathbf{0}$ to ensure energy dissipation of the structural system under applied loads.
In case of associate flow rule, the vector of generalized strains is normal to the yield function **y.** In this case, it is easy to show that:

$$\mathbf{V} = \mathbf{N} \tag{17}$$

The definition of the plastic behaviour of joints is completed by the additional condition:

$$\mathbf{y}^T \mathbf{z} = 0 \tag{18}$$

known in plasticity as complementarity relation, which allows to have positive components of plastic multipliers **z** only when the stress state is on a yield plane.

## 3. FORMULATION OF THE LIMIT ANALYSIS PROBLEM AND SOLUTION PROCEDURE

It is well known that, under the hypothesis of classical plastic theory, including fully associate plastic flow rules and convex yield functions, the lower and upper bound formulations of the limit analysis problem lead to two dual linear programming problems, static and kinematic, whose unique solution is the load factor α.
On the basis of previous assumptions, the formulation by linear programming of the static theorem of limit analysis, stating that the collapse load corresponds to the maximum load factor associated to a static admissible distribution of internal forces satisfying yield conditions, is then:

$$\begin{aligned} &\max && \alpha \\ &\text{subject to:} && \mathbf{A} \cdot \mathbf{x} = \mathbf{f}_D + \alpha \mathbf{f}_L \\ & && \mathbf{N}^T \cdot \mathbf{x} \le \mathbf{r} \end{aligned} \tag{19}$$

In the present study, to take into account the nonassociative behaviour in sliding, the iterative solution procedure proposed by Gilbert et al. [5] was used to find the minimum collapse load, instead of solving the underlying mixed complementarity program.
Moreover, a specific iterative procedure was used to take into account, in a simple way, the brittle behaviour of mortar joints and unit interfaces undergoing cracking failure. The procedure is based on a different plastic behaviour of uncracked and cracked interfaces and it is organized in the following steps: 1. Limit analysis of the rigid block model with effective values of plastic strength based on yield functions related to uncracked behaviour; 2. Detection of contacts undergoing cracking failure; 3. Redefinition of mechanical properties at the selected interfaces according to cracked behaviour; 4. Repetition of steps from 1 to 3 until convergence.

## 4. APPLICATION TO A CONFINED SHEAR WALL EXAMPLE

A computer program was developed on the basis of proposed formulation and applied to the limit analysis of the masonry panels with openings tested by Raijmakers and Vermeltfoort [6].
The shear walls have been investigated by various authors, using different modelling methods and assumptions on loading and boundary conditions. As for limit analysis, the shear walls have been investigated by Sutcliffe et al. [7] and Milani et al. [8] as well, on the basis of a finite element heterogeneous and homogeneous approach.
The panels considered in this section are the specimens labelled as J4D and J5D in [6]. The dimensions of the shear walls are 990 × 1000 mm. The panels were made of solid clay bricks with dimensions of 210 × 52 × 100 mm and assembled with 10 mm thick mortar joints. The panel was tested under a uniform preload of 30 kN and then applying a horizontal monotonic force with restrained rotation and vertical displacement.
The rigid block model generated for limit analysis and the schematization of the loading protocol are depicted in Figure 3. Also in this case, a block model with rigid masonry units and tensionless joint interfaces was developed for comparisons.

**Table 1** Interface mechanical properties for the confined shear wall J4D and J5D

| Joints | | | | Units | | | |
|---|---|---|---|---|---|---|---|
| $f_{tef}$ [N/mm$^2$] | $f_{cef}$ [N/mm$^2$] | $\mu$ | $c_{ef}$ [N/mm$^2$] | $f_{tef}$ [N/mm$^2$] | $f_{cef}$ [N/mm$^2$] | $\mu$ | $c_{ef}$ [N/mm$^2$] |
| $\nu_t \cdot 0.25$ | $\nu_c \cdot 10.5$ | 0.75 | 1.4 $f_{tef}$ | $\nu_t \cdot 1.4$ | $\nu_c \cdot 15$ | 1.0 | 1.4 $f_{tef}$ |

The values of mechanical properties for contact interfaces are reported in Table 1. The effectiveness factors have been determined according to Eq. 9 and Eq. 10
Similarly to Orduña and Lourenço [1], in this study the experimental values of the vertical peak load measured in the tests by Raijmakers and Vermeltfoort were used to evaluate the lateral collapse load.
As for boundary conditions, in order to take into account the effective stress distribution of normal and shear stresses at the top of the panel, in this study a modified version of the general procedure developed by Orduña & Lourenço for the analysis of confined shear masonry walls was used [1].
The modified procedure proposed herein is based on a direct definition of rotational boundary condition, thus avoiding iterative analysis for the determination of the vertical force location. For the formulation of the limit analysis problem presented in previous sections, the proper definition of the boundary conditions corresponding to the actual stress distribution at collapse in case of confined panels is straightforward and is based on the appropriate expression of equilibrium equations of the top beam.
The results of the analysis are shown in Figure 4. Comparison of numerical and experimental failure loads is reported in Table 2.
The failure mechanism obtained for the rigid block model with cracking units is diagonal shear failure. The predicted collapse load is 49.5 kN in a good agreement with experimental results.
The comparison with the results obtained with other modelling approaches in the literature (see Tab. 2) shows the accuracy of the implemented formulation in predicting the structural behaviour of the selected case study.

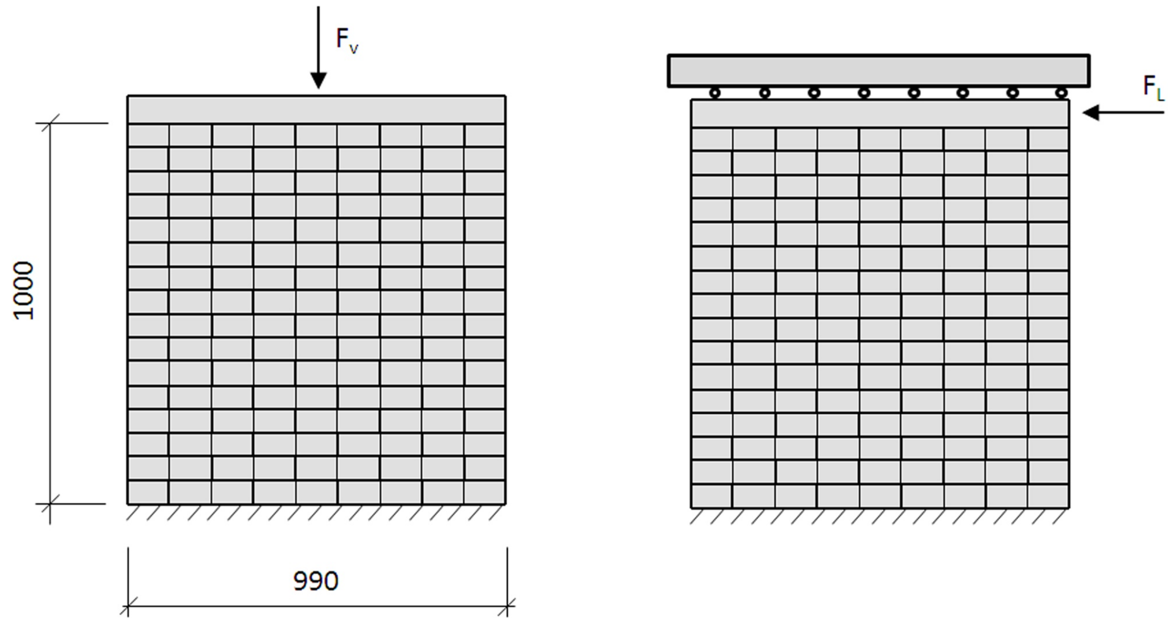


**Fig. 3** Rigid block model of confined shear walls J4D and J5D. Dimensions and loading protocol

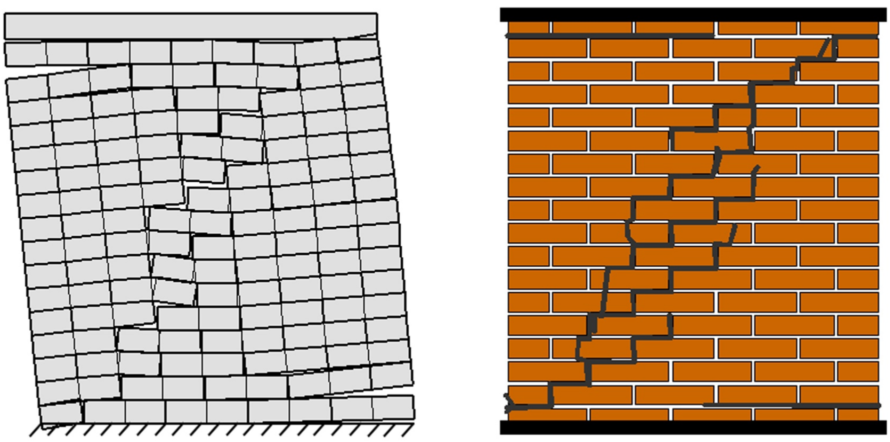

**Fig. 4** Shear walls J4D and J5D: Comparison of computed failure mode and experimental failure pattern

**Table 2** Confined shear wall J4D and J5D: Comparison of experimental and rigid block limit analysis collapse loads

| Specimen | $F_v$ [kN] | $F_L$ [kN] | | | |
|---|---|---|---|---|---|
| | | Experimental | Model with cracking units and cohesive joint interfaces | Heterogenous approach. [7] | Homogeneous Approach [8] |
| J4D | 30 | 49.9 | 49.5 | 26.5 | 24.2 |
| J5D | | 52.8 | | | |

## 5. CONCLUSIONS

A rigid block model with cracking units has been developed for limit analysis of masonry shear walls under in-plane loads based on a micro-modelling approach. Masonry panels are discretized with two rigid blocks per masonry unit divided by a vertical interface. Failure is concentrated at mortar and masonry units contact interfaces and includes cracking, crushing and sliding with non associative flow rule. The corresponding mathematical problem is solved by iterative linear programming.

To evaluate the accuracy of the proposed formulation, a validation study was carried out and an application to a benchmark problem from the literature were presented.

The results of the numerical validation analysis were in perfect accordance with other limit analysis formulation approaches based on the direct definition of the cracking units in the model. Comparison with experimental tests showed a good agreement in terms of collapse load and mechanism.

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