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LIMIT ANALYSIS OF THE DUOMO DI MILANO VAULTING SYSTEM

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Abstract. *The present research was carried out with the aim of understanding the structural configuration of the Duomo di Milano, in relation to some of its particular features, such as a double vaulting system and iron ties at the support of Gothic arches. Being one of the most remarkable heritage structures, a multidisciplinary approach was employed for dealing with the Duomo di Milano, combining several fields of engineering expertise. Its structural history has been analysed trough different building schemes with possible causes for past interventions. The geometry of the Cathedral's vaults was derived from recent laser on-site measurements and the available literature on past surveys. Beside the detailed description of the geometry, the materials layout is also given with their main characteristics. Hypotheses were made on the construction methods, possible scaffolding and framework used for the erection of vaults. Limit analysis theory was employed for analysing the structural behaviour of the vaults and finding load paths inside the masonry elements. Within the aim of assessing the individual element stability and its role in the overall safety of the structure, the performed analysis presents an initial step. The results of the present analytical approach might be compared to the ones obtained by a sophisticated finite element analysis in the future.*

1 INTRODUCTION

The main nave vaults of Duomo di Milano have a complex structural system and their assessment requires knowledge of design origins, geometrical properties, historical evolution and the constructive stages of the structure. Neglecting one of these steps could lead to a model which would give non-realistic results, or represent different structure. Structural analysis by means of limit analysis is a part of a multidisciplinary assessment project for this remarkable monumental building. The first objective is to understand the structural behaviour by finding possible states of equilibrium which by the Safe Theorem [1] would mean that the structure is safe and stable. In relation to the usual second task in the assessment of existing structures, that is the understanding of origins and consequences of crack pattern in masonry, during the present research one should understand in which way the vaults of the main nave have not suffered any mayor cracks in the past. Once the load paths are found inside the masonry elements, equilibrium schemes could be worked out at the top of the piers resulting from vaults and arches. These are supposed to be balanced by a combined action of ties and lateral walls, but the amount of the thrust resisted by each of these elements is unknown, so several possible configurations for the portion resisted by the ties should be assessed and compared with experimental and numerical results from other studies.

2 CONSTRUCTION HISTORY AND THE GEOMETRY OF DUOMO DI MILANO

The Duomo di Milano has a complex geometry characterized by a double-vaulting system in all of its five naves, probably unique among world-wide Gothic Cathedrals and a complex structural system. A recent study [2] on the structural history of its load-bearing system summarized the main factors such as construction history, previous restoration works, soil settlement and underground vibrations, which in the past and the present might influence its structural performance or cause damage. A detailed construction phase map was derived from historical investigation together with the results of visual inspection and preliminary non-destructive dynamic tests which confirmed that the ties are currently active in resisting lateral thrust [3].

The erection of each bay started from the extreme lateral naves followed by the inner lateral naves, and the main nave was built at the end. The construction of the main nave itself probably started with construction of stone masonry piers in their entire height, placing on them the *tas-de-charge* – single block of marble stone (similar to the one reported in [1], page 107) which provides the base for the diagonal ribs, transversal arches and filling. The vaults in this way might have been constructed without any formwork, apart from that supporting arches and ribs. The wrought iron ties were placed on the top of these single blocks at the height of 34.5m from the floor level, and approximately at that point the diagonal rib and Gothic arch made of multiple *voussoirs* develop separately, built probably over a temporary wooden centering [4]. The space between the springing of the ribs and the longitudinal wall was filled with brick masonry parallel to the rib erection, providing the structure through which the thrust line from the vault can pass. Subsequently this backing cone was the solid base for the further construction of the vault over the arches and ribs starting from the top of the cone infill and, as the Authors believe, without formwork for the 0.4m thick masonry vault. Once the cross-vault was completely constructed, spandrels were built over the Gothic arches as the basis for the barrel vault (0.4m thick) and a longitudinal beam (width 3.2m), all made of masonry, providing a support for the marble roof to be built on the top (Figure 1).

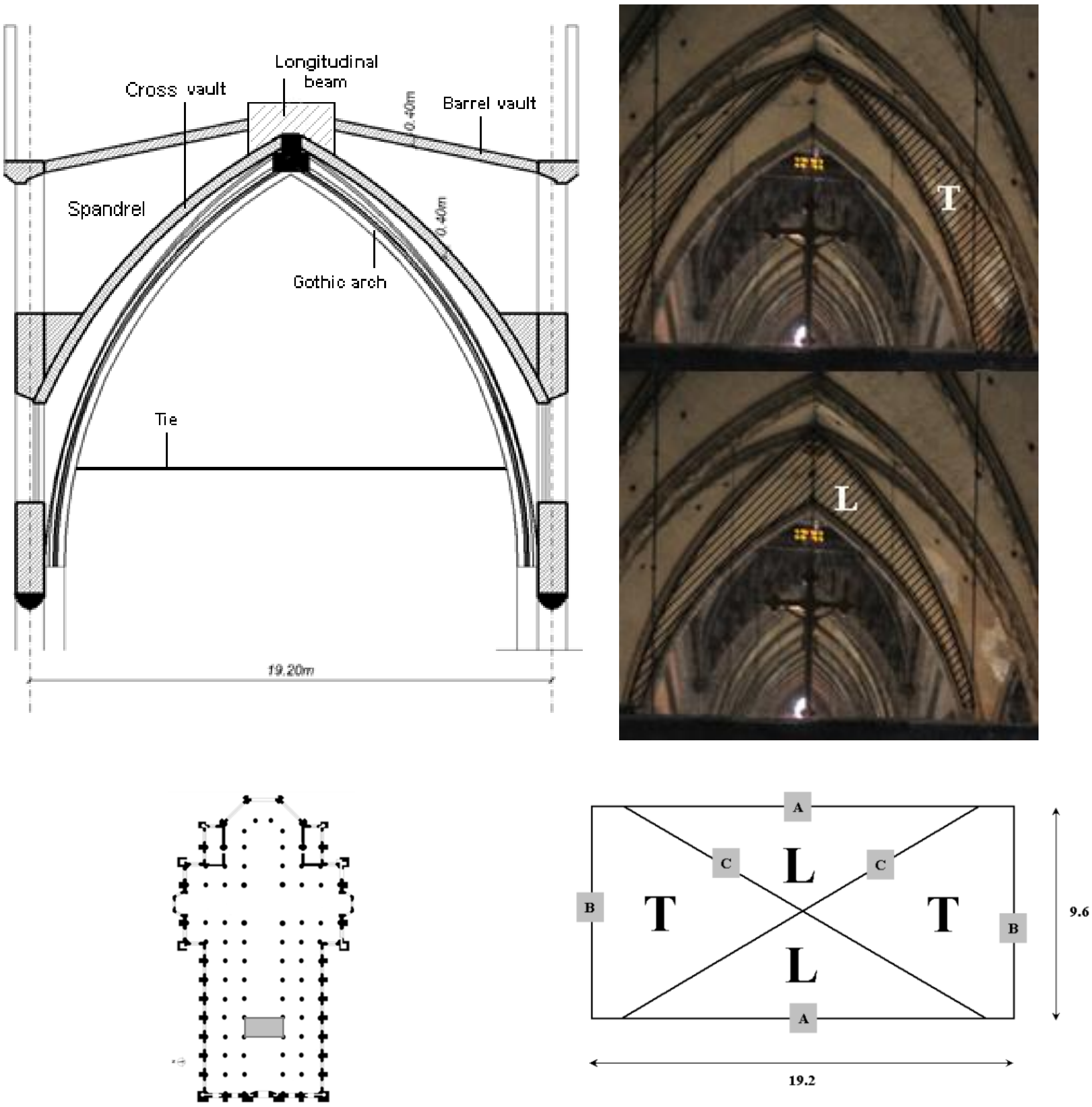

Figure 1 : Geometry of the main nave vault.

Transversal (T) and longitudinal (L) vaults can be distinguished in the main nave vault (Figure 1), bearing on Gothic (transversal) arches, longitudinal walls and diagonal ribs (A, B and C in Figure 1, respectively). In the present case study, the transversal vault has the shape of a truncated incomplete dome, providing lateral forces to the masonry in the longitudinal vault which is acting as a sort of buttressing flat arch system [1]. From the layout of masonry unit through the vault observed during the on-site visual inspection (Figure 3), it is possible to distinguish courses and estimate dimensions of the bricks (when compared to the known dimension of the rib) as approximately 0.15m x 0.15m x 0.4m. Although the courses are visible only in some parts of few transverse vault compartments as a consequence of moisture-caused deterioration, it is clear that its each second row is made of courses placed in horizontal parallel planes rotated by 90° with respect to the previous row, ensuring in this way the monolithic behaviour through the vault thickness and facilitating the construction process. This is complementary with the assumption of the construction process and building part of the main nave vault between the level of infill and the starting angle of embrace from the crown as a self-standing incomplete dome. In addition, estimated equilibrium forces in the present work are to be included in future structural analysis of the Cathedral's representative bay and to be compared with the results from Finite Element Method (FEM) model.

The Fabbrica del Duomo, the institution that built and is taking care of the Cathedral, possesses extensive documentation on dimensions and constructive details. It was nonetheless essential to perform an on-site visual inspection and verify the existing documentation by direct measurements where possible. The Main nave vault of Duomo di Milano is covering a rectangular base with dimensions 19.20m x 9.60m and has a free span of 16.65m (Figure 1). The transversal arches and diagonal ribs of the main nave have nearly rectangular cross section with nominal dimensions 1.05m x 0.57m and 0.68m x 0.52m, respectively. They are Gothic pointed arches with radius of 16.65m, composed of multiple "Candoglia" marble *voussoirs* (Figure 2). The geometry of the transversal vault (Figure 1), between the two ribs and the longitudinal wall, is formed by a rotation of the diagonal rib around a vertical axis in the centre of the nave (point O). The resulting shape is a pointed segmental dome, with centre in point Or (Figure 3), radius of the middle plain R=16.85m, thickness 0.4m, angle of opening θ=62°, starting angle of embrace from the crown φ_0=30° and the total angle of embrace between the crown and the level of infill φ=55°.

No major cracks are visible on the inside of the vault, while on the outside is present a concrete layer reinforced with a steel mesh constructed during the 1970's restoration works, currently without observable cracks. Detachment of mortar, humidity and efflorescence are observable on the inside of several vaults nowadays and its causes should be investigated more in depth. The material characteristics (Table 1) were determined during the restoration in 1980's, except for the ties, which were characterized during the current research at Politecnico di Milano laboratory for Materials Structures and Constructions in 2011.

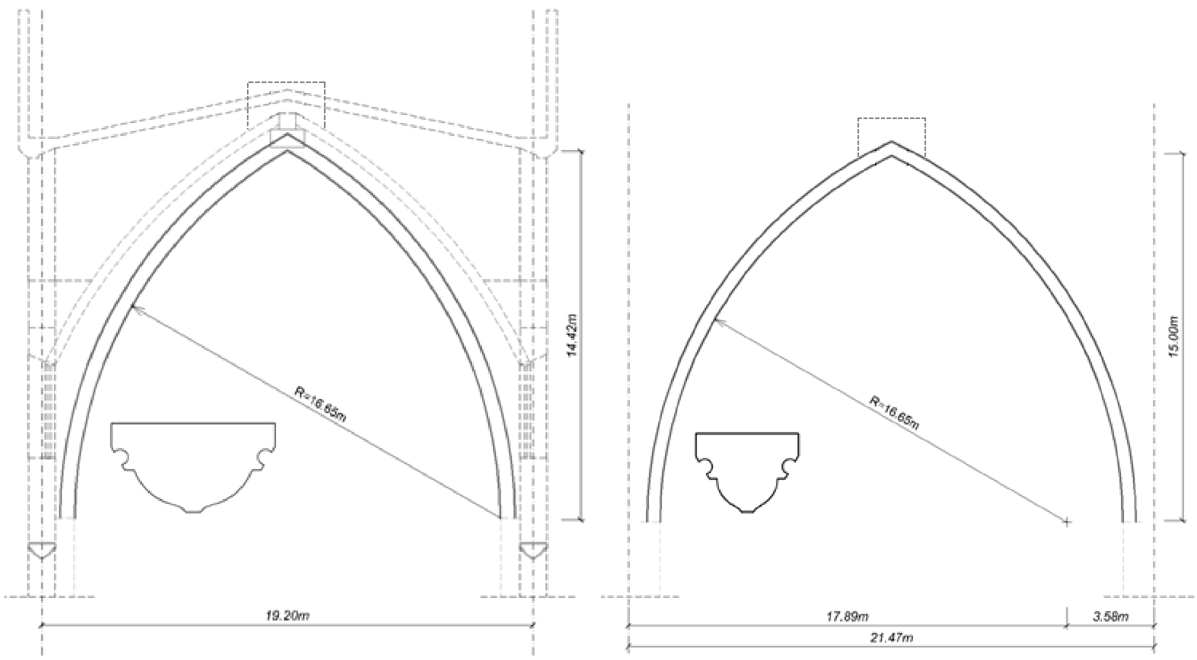

Figure 2 : Geometry of the main nave's transversal arch (left) and the diagonal rib in its own plane (right).

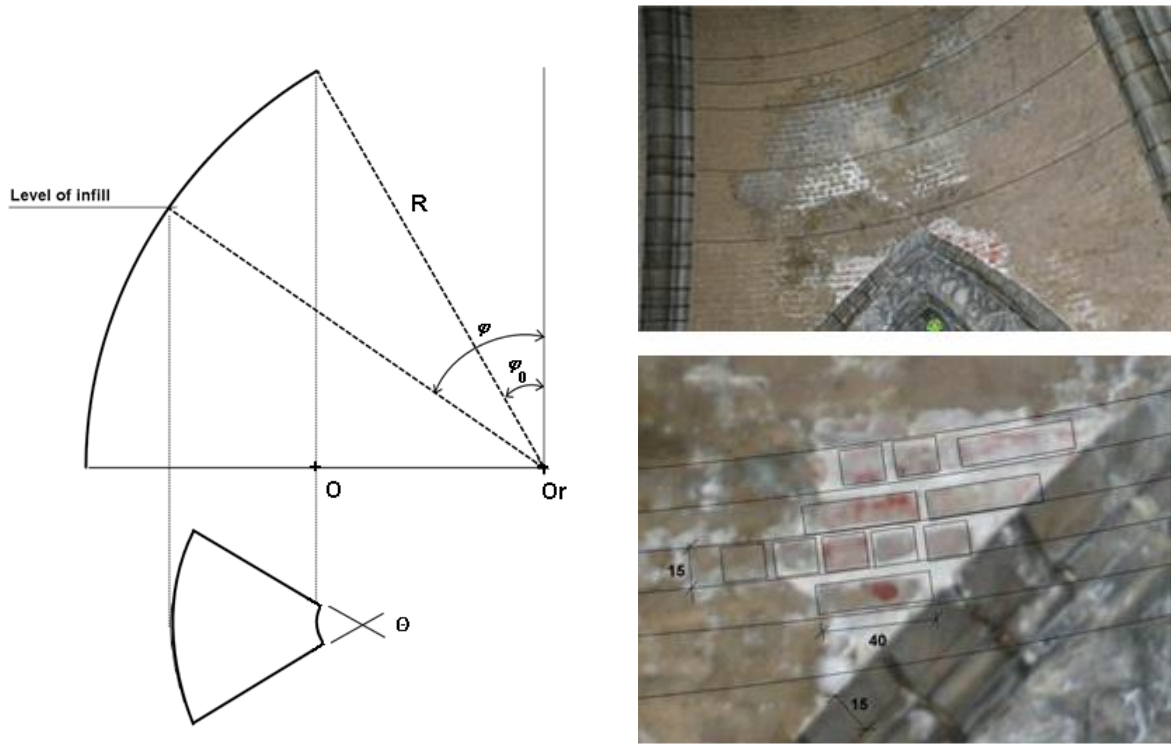

Figure 3 : Transversal vault: geometry of the pointed truncated segmental dome (left) and detail of brick units (right).

Table 1 : Material properties.

Structural element	Material	Weight [kg/m^3]
Gothic arch Cross ribs Arches of lateral naves Keystone	Marble "Candoglia"	2750
Spandrel above transverse arch Main nave vault Longitudinal beam over the main nave	Brick masonry	1800
Foundations	Mixed masonry	2200
Infill of vaults	Diverse materials	1800
Ties	Wrought iron	7792

3 STRUCTURAL ANALYSIS

The main objective of analysing vaulting system of Duomo di Milano is to study the possible states of equilibrium and in this way understand its structural behaviour [5, 6]. The aim is to estimate the total value of the lateral thrust from vaults and ribs during current work, which would during later developments be combined with the one from arches. On the top of the piers, these forces are resisted by a combined action of metallic ties and walls and its estimate will in future be compared to the one measured experimentally or predicted by FEM calculations. It should be noted that the load path worked out during this research is just one of the possible solutions and according to the master “safe” theorem [1], when at least one solution for which the line of thrust lying within the masonry can be found, the structure is safe – if the assumptions of the theory are verified. A preliminary visual examination indicated that there are no eye-visible cracks on the brick masonry of the main nave vaults. Structural elements were identified analysing the constructive history of the Cathedral, on-site inspection and particular constructive features. Based on the considerations made in the previous sections on building vaults as self-supporting parts on already built ribs, arches and walls, and on the current stage without visible cracking of masonry, one of the possible structural schemes could be a membrane behaviour in the transversal vault of the main nave and “flat arch” [1] behaviour of the longitudinal vault (Figure 4 and Figure 5).

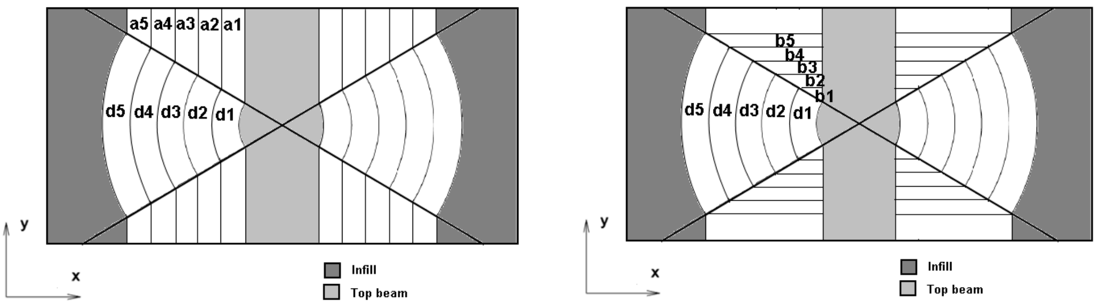

Figure 4 : Main nave vault adopted division in the horizontal plane.

(a)

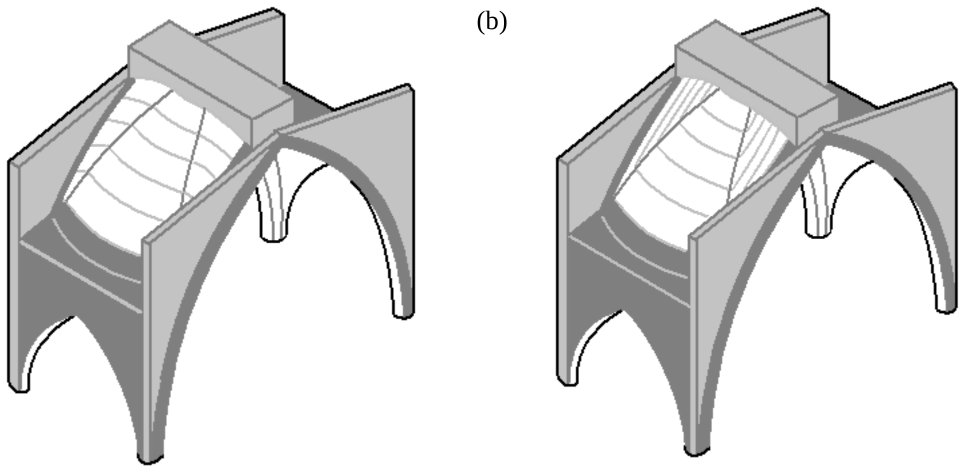

Figure 5 : Main nave vault adopted division: Longitudinal vault sliced as longitudinal arches (a) and transversal arches (b).

Considering the geometry of barrel vaults above the main nave vaults and the fact that they are bearing on supporting walls and Gothic arches, they are assumed to be the dead load and taken into account while estimating the total force in ties, which is not the topic of the present paper. At the moment, it is not possible to have clear image if and how deep the stone *voussoirs* penetrate into the 0.4m thick masonry vault. However, during the past restoration works it was concluded that in most of the vaults, the diagonal ribs were immerged into the masonry vault [7], which could be assumed from pieces of stones going out from the vault volume in correspondence of the diagonal ribs on the outer side of the vault. Therefore, the assumption in the present research is that the diagonal ribs are interacting with the masonry vault and are resisting forces from the vault.

3.1 Transversal vault

The transversal vault of the main nave is assumed to behave as a membrane since no mayor cracking is visible and its geometry is partial dome of revolution. It is sliced in 5 elements as shown in the Figure 5 using horizontal planes. The very first slice above the fill level (if taken into account would be 6^{th} in the present work) was not taken into account since the corresponding part of the longitudinal vault is too short to resist any force and it is assumed that the forces form this part of transversal vault are directly transferred to the transversal walls and further on to other naves.

Based on the geometry of the transversal vault, its structural history and current non-cracked stage, the transversal vault is assumed to be a thin shell, with resulting membrane forces: self-weight of each element (W), hoop force (P), top (F_t) and bottom meridian forces (F_b). Guastavino, Jr. worked out force polygons for thin masonry domes [8] and showed that the top (F_t) and bottom meridian forces (F_b) are equilibrated by the self-weight of each element (W) and the component of hoop force in x-direction (C). This can be applied to the present case, with difference when analysing the incomplete dome, that the other component in y-direction (T) should be resisted by other elements. Therefore, in present work the resultant of the hoop force in y-direction (T) is further applied as a load on the longitudinal vault. The current procedure for calculating resulting forces included:

- Calculating the W from the geometry of sliced dome (Table 2) (Figure 6a).
- Formation of force polygons for each segment *i*: force C is assumed to be horizontal, while angles of forces F_t and F_b are taken as orthogonal to each segment - following the classical theory of shells [9] (Figure 6b and c).
- Measuring forces C from the force polygons and decomposition to forces P and T, following the geometry of the segmental dome with angle of opening θ=62° (Figure 6d).
- Imposing forces C/2 and T as load to transversally and longitudinally sliced longitudinal vault, respectively, taking into account 50% of the vault weight in each direction of slicing.

Since the transversal vault has the shape of a truncated segmental dome, the first element is loaded with distributed load, coming from the top beam, resulting in force F_{tb}=29.36kN (sum of 22.60kN as part of the longitudinal beam weight and 6.76kN as ¼ weight of the keystone). Resulting forces for each element (Table 2) and polygons of forces (Figure 6) show that each element is equilibrated in vertical plane by forces F_t, F_b, W, C and compressed, assuming that the remaining forces Ti are resisted by the longitudinal vault, investigated as next step.

Table 2 : Geometry and resulting forces of the transversal (domical) vault.

i	r [m]	W [kN]	F_t [kN]	F_b [kN]	C [kN]	T [kN]	P [kN]
1	2.09	17.34	61.38	85.63	17.80	15.10	17.53
2	3.34	34.71	85.63	132.94	33.30	28.20	32.75
3	4.58	47.57	132.94	189.84	34.20	29.00	33.67
4	5.82	72.52	189.84	269.87	40.20	34.10	39.58
5	7.06	87.96	269.87	355.21	28.40	24.10	27.97

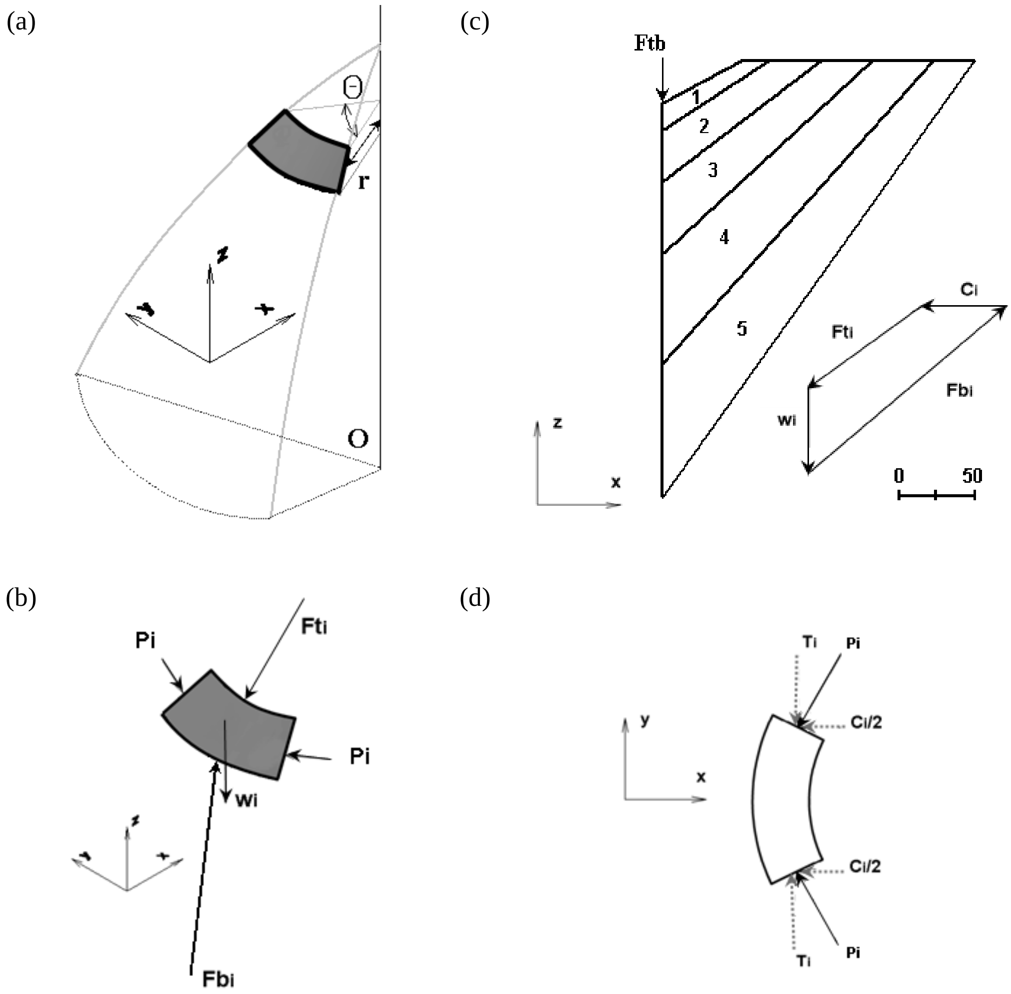

Figure 6 : 3D Equilibrium of forces for domical element *i*.

3.2 Longitudinal vault

The longitudinal vault is characterized by the two main curvatures, the line of the Gothic arch and the curvature of the transversal cross section of the vault in its crown. Laser measurements confirmed that its surfaces are formed by the translation of the Gothic arch curvature following the transversal crown curvature. As explained in the previous section, forces resulting from the membrane theory analysis of transversal vault should be further resisted by the longitudinal vault. With this aim, the longitudinal vault was sliced in two series of arches, transversally and longitudinally, each attributed with 50% of the total vault weight.

Longitudinal slices are assumed to work as series of active flat arches (Figure 6a), loaded with horizontal forces T, coming from corresponding element *i* in the transversal vault. This is one of the possible solutions, providing the active resultant in the flat arch with resultants V_1 and V_2 and was chosen by placing the thrust action in the middle of element's thickness, corresponding to the point where the forces T_i are acting (more details on active thrust in flat arch can be found in [1]). In all elements, the line of thrust was inside the geometry of the element, ensuring the stability of the element. In this way, the longitudinal vault is equilibrating the transversal one. Forces V_1 and V_2 are later on transferred to the diagonal ribs and Gothic arch, respectively.

Transversal slices are assumed to work as series of classical arches (Figure 6b), loaded on the top with concentrated load coming from longitudinal masonry beam (F_i). The thrust line position was assumed in the middle of the cross-section, corresponding to the position of forces in the middle plane of domical vault. Vertical reactions V_i and horizontal H_i were estimated (Table 3) and further on are summed with forces $V_{1,i}$ and $C/2_{,i}$, respectively and transferred to the diagonal rib.

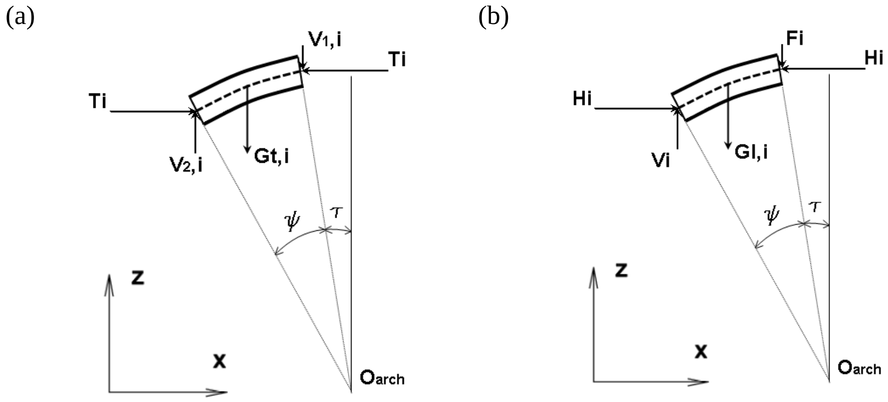

Figure 7 : Forces in vertical plane for the longitudinal slices (a) and transversal slices (b) of the longitudinal vault.

Table 3 : Geometry and resulting forces of the longitudinal slices.

i	R_a [m]	τ [°]	ψ [°]	T [kN]	G_t [kN]	V_1 [kN]	V_2 [kN]
1	9.48	9	19	15.10	10.25	0.00	10.25
2	9.16	13	16	28.20	9.23	6.29	15.53
3	9.19	17	13	29.00	7.53	8.91	16.44
4	10.59	24	9	34.10	5.99	15.57	21.55
5	18.59	32	3	24.10	3.47	14.23	17.70

Table 4 :Geometry and resulting forces of the transversal slices.

i	R_a [m]	τ [°]	ψ [°]	H [kN]	G_l [kN]	F [kN]	V_i [kN]
1	19.12	33	3	14.17	2.04	8.82	10.86
2	19.19	33	6	16.97	3.73	10.49	14.22
3	19.22	33	10	20.06	6.23	12.63	18.85
4	18.25	33	14	23.12	8.73	15.18	23.91
5	18.63	33	19	29.29	11.46	18.10	29.56

3.3 The diagonal rib

The diagonal rib of the main nave was divided in 5 elements, corresponding to the 5 slices of each vault and in its vertical plane is loaded with forces Y_i:

$$Yi = G_{R,i} + +V_{1,i} + V_i \quad (1)$$

where the force $G_{R,i}$ is the self-weight of each element of the rib, $V_{1,i}$ is one of the vertical reactions of longitudinally sliced longitudinal vault and V_i is the vertical reaction of transversally sliced longitudinal vault. In its horizontal plane it is loaded with forces h_i and s_i:

$$h_i = \cos\frac{\theta}{2} * \left(H_i - \frac{C_i}{2}\right) \quad (2)$$

$$s_i = \sin\frac{\theta}{2} * \left(H_i - \frac{C_i}{2}\right) \quad (3)$$

which are the resultants in the local coordinate system of the rib, coming from the horizontal reaction of transversally sliced longitudinal vault H_i and the horizontal force of the domical vault $C_i/2$. The rib is also loaded in its crown with a concentrated force F_r=12kN (sum of 5.24kN as part of the longitudinal beam weight and 6.76kN as ¼ weight of the keystone). Assuming the maximum thrust, reactions are calculated as H_r=106.39 and V_r=160.31. Line of the thrust results to be inside of the rib geometry, both in the vertical and horizontal planes, (Figure 8) which is one of the possible equilibrium states.

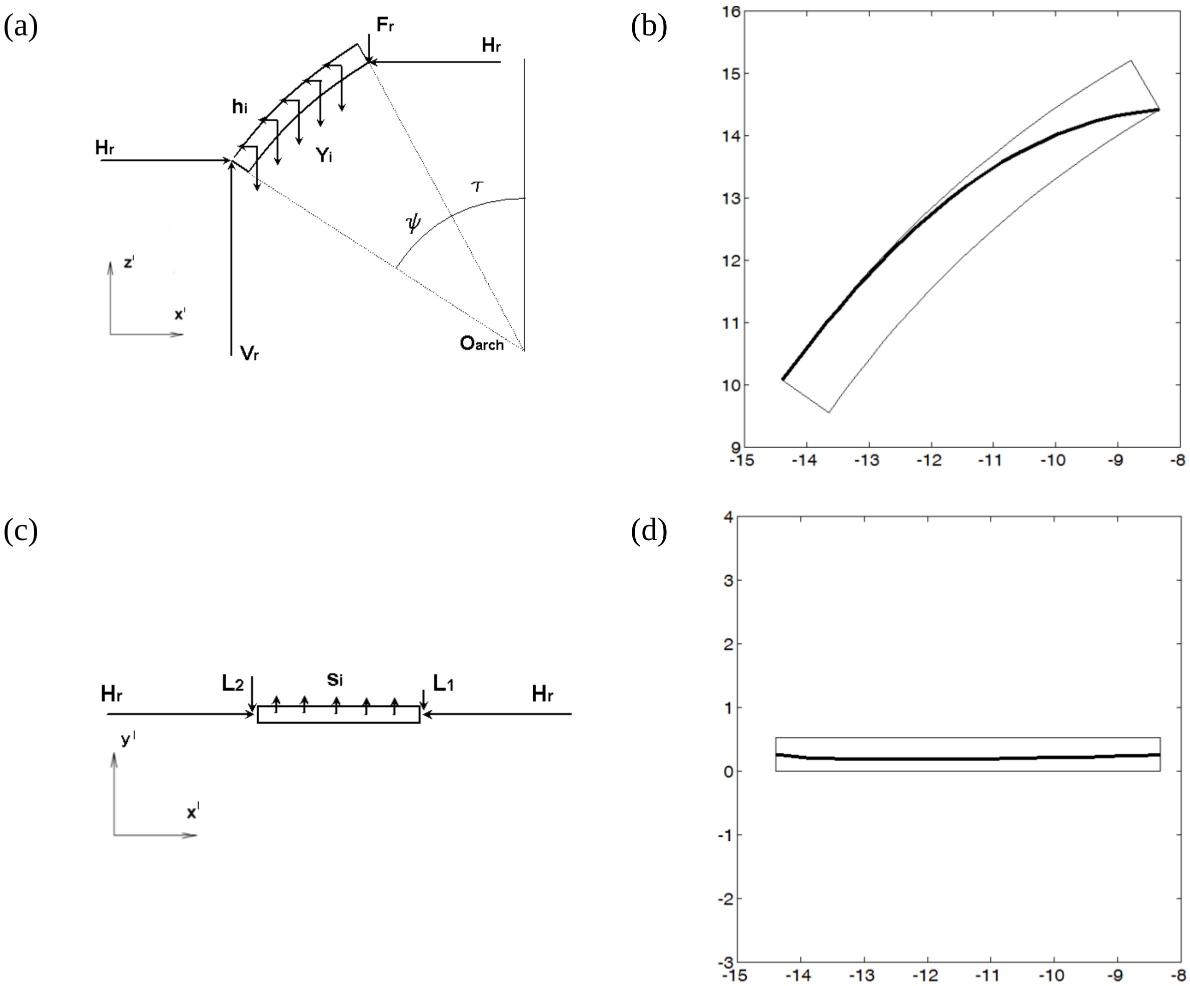

Figure 8 : Static of the rib in vertical (a and b) and horizontal (c and d) planes.

Table 5 : Load on the rib.

i	Y [m]	h [kN]	s [kN]
1	10.86	4.52	2.72
2	7.93	0.28	0.17
3	9.94	2.53	1.52
4	8.35	2.59	1.55
5	15.33	12.94	7.77

4 CONCLUSIONS

This work finds one of the possible equilibrium states for the main nave vault of the Duomo di Milano. Forces coming from this particular scheme are supposed to be transferred to the rest of the structure – arches and filling of the vault, participating in the global equilibrium of the Cathedral and estimation of the force in the wrought iron ties. The analytical solution in the current work was drawn from understanding of the structural history, geometry and the constructive features and resulted in finding possible equilibrium of the main nave vault of the Duomo di Milano. Structural analysis of such a complex structural parts, by means of simple equilibrium analysis can be later on compared with advanced numerical analysis, such as FEM, providing insights into the structural behaviour of the analysed elements and constructive process [10]. Future work will include finding equilibrium for the whole main nave vault and these results will be used as verification of the refined FEM model and experimental results in terms of the tie internal tension forces.

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